Lecture 9 1/25/2021

Last time.

Given $f(x) = \sum_{i=0}^{n} b_i x^i \in F[x]$ and $A \in M_n(F)$ we have $f(A) = \sum_{i=0}^{n} b_i A^i$ $(A^0 = I)$

Also for $\phi \in End_F(V)$ $\dim_F V = n$

$$f(\phi) = \sum_{i=0}^{n} b_i \phi^i \quad (\phi^0 = 1_V)$$
$$\in End_F(V)$$

minpoly$(A) = g \in F[x]$ monic

s.t. $(g) = \ker \varepsilon_A$ $\quad \varepsilon_A : f(x) \mapsto f(A)$

minpoly$(\phi) =$ " " "

" " $\ker \varepsilon_\phi$ $\quad \varepsilon_\phi : f(x) \to f(\phi)$

minpoly$(\phi) =$ minpoly$(M_\beta^\beta(\phi))$

for any basis β.

Prop. Let $\phi : V \to V$ be in $\mathrm{End}_F(V)$ where V is a f.d. F-space.
Let V be an $F[x]$-module with $x \cdot v = \phi(v)$.
Let $f_1, f_2, \ldots f_n$ be inv. factors.

Then $\mathrm{minpoly}(\phi) = f_n$.

Pf.

$$V \underset{F[x]}{\cong} F[x]/(f_1) \oplus \cdots \oplus F[x]/(f_n)$$

$$f_1 | f_2 | f_3 \cdots | f_n.$$

Look at $\mathrm{Ann}_{F[x]} V$.

$$\mathrm{Ann}_{F[x]} F[x]/(f_1) = (f_1)$$

Then

$$\mathrm{Ann}_{F[x]} V = (f_1) \cap \cdots \cap (f_n)$$

But $(f_1) \supseteq (f_2) \dots \supseteq (f_n)$

So $\mathrm{ann}_{F[x]} V = (f_n)$.

Now claim $\mathrm{minpoly}(\phi) = \mathrm{Ann}_{F[x]} V$.

Check for $h \in F[x]$. $h = \sum_{i=0}^{m} a_i x^i$

$$
\begin{aligned}
& h \cdot v \\
&= \sum_{i=0}^{m} a_i x^i \cdot v \\
&= \underbrace{\sum_{i=0}^{m} a_i \phi^i(v)} \\
&= h(\phi)(v)
\end{aligned}
$$

So $h \cdot v = 0 \quad \forall v \in V$

iff $h(\phi)(v) = 0 \quad \forall v \in V$

iff $h(\phi) = 0$.

So $(h_n) = \mathrm{Ker}\, \varepsilon_\phi$

So $h_n = \mathrm{minpoly}(\phi)$.

<u>Thm.</u> (Cayley–Hamilton). Let $A \in M_n(F)$. Let $f = \text{charpoly}(A)$. Then $f(A) = 0$.

<u>Pf.</u> Let $V = F^n$, as an $F[x]$-module where x acts as $\underline{A}$. The invariant factors are $f_1, \dots, f_n$.

We showed

$$\text{charpoly}(A) = f_1 f_2 \cdots f_n$$

$$\text{minpoly}(A) = f_n.$$

So $\text{minpoly}(A) \mid \text{charpoly}(A)$.

So if $g = \text{charpoly}$ then

$g(A) = 0$.

<u>Rmk.</u> We also have if

p is prime (irreducible)

in $F[x]$ and $p \mid \text{charpoly}(A)$

then $p \mid \text{minpoly}(A)$.

Since if $p \mid f_1 f_2 \cdots f_n$

then $p \mid f_i$ for some i

So $p \mid f_n$.

So $p \mid \text{minpoly}(A)$.

Ex. $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

find minpoly(A)

charpoly(A)

rational, Jordan form.

A upper triangular.

so charpoly(A)

$$= \det\begin{pmatrix} x-2 & & * \\ & x-1 & \\ 0 & & x-2 \end{pmatrix}$$

$$= (x-2)^2(x-1)$$

Now minpoly(A) involves every prime in charpoly(A)

$= (x-2)(x-1)$ or

$(x-2)^2(x-1)$.

Check:

$(A-2I)(A-I) =$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 0.$$

So minpoly$(A) = (x-2)(x-1)$.

So inv. factors are

$f_1 = (x-2) \quad f_2 = (x-2)(x-1)$

$$\shortparallel$$

$$x^2 - 3x + 2$$

rational form

$$\left[\begin{array}{c|cc} 2 & & \\ \hline & 0 & -2 \\ & 1 & 3 \end{array}\right]$$

elementary divisors

$(x-2), (x-2), (x-1)$.

Jordan form

$$\begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}$$

Generalized eigenspaces.

F alg. closed.

V F-space $\phi \in \mathrm{End}_F(V)$

$\dim_F V = n$.

<u>Def.</u> $v \in V$ is a generalized eigenvector

with eigenvalue λ if

$$(\phi - \lambda 1_V)^n (v) = 0.$$

[i.e. $(x-\lambda)^n \cdot v = 0$

for associated

$F[x]$ - module]

If $(\phi - \lambda 1_V)(v) = 0$

then $\phi(v) = \lambda v$.

v is an eigenvector.

$$(\phi - \lambda 1_V)^2 (w) = 0$$

$(\varphi - \lambda 1_V)(w)$

$= \varphi(w) - \lambda(w)$

is an eigenvector.

Define

$V_\lambda = \left\{ v \in V \;\middle|\; \begin{array}{c} v \text{ is a} \\ \text{generalized} \\ \text{eigenvector} \\ \text{for } \lambda \end{array} \right\}$

is a subspace of V.

$= (x-\lambda)$-primary component of V.

So $V \cong V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_m}$

for $\lambda_1, \dots, \lambda_m$ distinct

Also

$$V_\lambda \cong F[x]/(x-\lambda)^{e_1} \oplus \cdots \oplus F[x]/(x-\lambda)^{e_s}$$

some $e_1, \dots, e_s$.

<u>Recall</u>: the number of

e_i s.t. $e_i \geq b$

$=$ F-dimension of

$V_\lambda[b] / V_\lambda[b-1]$

<u>$V_\lambda[b]$</u> $= \{ v \in V_\lambda \mid (x-\lambda)^b \cdot v = 0 \}$

Ex.

$$A = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & & & \\ & & \lambda & 1 & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

$V = F^5$ acted on by A.

elementary divisors

$(x-\lambda)^2 \quad (x-\lambda)^3$

$V = V_\lambda$

λ eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$= \dim V_\lambda[1] / V_\lambda[0]$

$$= \{ v \mid (x-\lambda)\cdot v = 0 \}.$$

basis for $\underline{\begin{array}{c}0\\1\\\hline 0\\0\\0\end{array}}$ $\underline{\begin{array}{c}0\\0\\\hline 0\\1\\0\end{array}}$

$V_\lambda(2)/V_\lambda(1)$

$$= \frac{\{ v \mid (x-\lambda)^2\cdot v = 0 \}}{\{ v \mid (x-\lambda)\cdot v = 0 \}}.$$

and $\begin{array}{c}0\\0\\\hline 0\\0\\1\end{array}$

basis for $V_\lambda(3)/V_\lambda(2)$.

Ex.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & 2 \end{pmatrix}$$

Find Jordan form.

Check $\mathrm{Charpoly}(A) = (X-1)^4$.

elementary divisors

$(x-1)^{e_1}, (x-1)^{e_2}, (x-1)^{e_3} \ldots$

$$V_1[b] = \{ v \in V \mid (A-I)^b \cdot v = 0 \}.$$

$$A - I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -2 & -1 & 1 \\ -2 & 0 & -1 & 1 \end{pmatrix}$$

dim Nullspace $= 4 - \text{rank}$.

$= 4 - 2 = 2$.

$$\underline{\underline{(A-I)^2}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

Nullspace has dim 3

$3 - 2 = \#$ elementary divisors, $(x-1)^e$

$e \geq 2$.

Only possibility is $(x-1)^3, (x-1)$.

Jordan form =

$$\left(\begin{array}{ccc|c} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & \\ \hline & & & \boxed{1} \end{array}\right)$$