Math 200b Final, March 14, 2011

You are not expected to do all problems. Do as many problems as you can, concentrating on the ones you know how to do well. All problems are weighted roughly equally. You may quote results proved in class or in the textbook. Generally you should avoid if possible quoting results proved only in homework exercises.

1a. Let F be a field of characteristic 0. Show if if A is an $m \times m$ matrix in $M_m(F)$ $(m \geq 1)$ such that $A^n = I$ for some $n \geq 1$, then A is diagonalizable (over the algebraic closure over F.)

1b. Give an example with justification which shows that for a field F of positive characteristic the result of (1a) need not hold.

2. Let M be a finitely generated nonzero torsion module over a PID R. Show that the following conditions on M are all equivalent:

(1) M is a cyclic R module;

(2) the list of invariant factors of M consists of a single nonzero element;

(3) The list of elementary divisors of M is of the form $p_1^{e_1}, p_2^{e_2}, \ldots, p_m^{e_m}$ for some distinct primes p_i and $e_i \geq 1$.

3. Let R be a commutative ring. Suppose that m and n are positive integers such that the free modules $M = R^m$ and $N = R^n$ are isomorphic as R-modules. Show that $m = n$. In other words prove that the rank of a free R-module is a well-defined concept.

(Note that R is not necessarily a PID, so you cannot apply the structure theory of modules over a PID. Suggestion: consider a maximal ideal I of R and look at the modules M/IM and N/IN.)

4a. Consider the splitting field K of the polynomial $x^3 - 5$ over $\mathbb{Q}$. Calculate the Galois group of $K/\mathbb{Q}$. Find all intermediate fields $\mathbb{Q} \subseteq E \subseteq K$, and determine which of these fields E are Galois over $\mathbb{Q}$.

4b. Find a primitive element for the extension $K/\mathbb{Q}$ (i.e. an element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$.)

5a. Let F be a field and let x, y, z be indeterminates. Consider

$$L = F(xz, yz, xy) \subseteq K = F(x, y, z).$$

Show that $[K : L]$ is finite and calculate its value.

5b. Prove that the subset $\{xz, yz, xy\}$ of K is algebraically independent over F. (Suggestion: use part (a).)

6. Let $F \subseteq K$ be a field extension, where $|F| = q < \infty$. Let $\alpha \in K$ be algebraic over F. Show that $[F(\alpha) : F] = d$, where $d = \min\{n > 0 | \alpha^{q^n} = \alpha\}$.

7a. Let K be a nonperfect field of characteristic p, so

$$L = K^p = \{a^p | a \in K\} \subsetneq K.$$

Show that $L \subseteq K$ is an inseparable extension.

7b. Show that the extension K/L has a primitive element if and only if $[K : L] = p$.

8. Recall that an algebraic extension $F \subseteq K$ is *normal* if whenever $f \in F[x]$ is an irreducible polynomial that has a root in K, f splits completely over K.

Consider an algebraic extension $F \subseteq K$ with $[K : F] < \infty$ which is normal (but not necessarily Galois.) Show that the following are equivalent for an intermediate subfield $F \subseteq E \subseteq K$: (i) E/F is normal; (ii) For all automorphisms $\sigma \in \mathrm{Aut}(K/F)$, $\sigma(E) \subseteq E$.