Math 200b Winter 2011 Homework 4

Due 2/4/2011 by 5pm in homework box

All exercise numbers refer to Dummit and Foote, 3rd edition. I like to list all exercises that I think are interesting. It is good to think about extra exercises if you can (at least look over the statements of the ones you don't have time to do.) However, only the exercises that are marked with a star are to be handed in for grading.

Reading assignment: 13.4-13.6

Section 13.1: #1, 4, 7

Section 13.2: #1, 4, 5*, 7, 8*, 10, 12, 13*, 14, 16*, 18, 19*, 20*, 21

Section 13.4: #1, 2, 3*, 4*, 5*

Here are some comments on 13.4 #5. First of all, your book does not carefully define what it means by a normal extension. It defines an algebraic extension $F \subseteq K$ to be normal if K is a splitting field over F for some collection of polynomials $f(x) \in F[x]$. However, it only defines the splitting field of a polynomial, not the splitting field of a collection of polynomials. The more general definition is what you expect: one says that K is the splitting field over F of some collection $\{f(x)\}$ of polynomials in $F[x]$ if every polynomial in the collection completely splits into linear factors in $K[x]$, and K is minimal with this property (equivalently, K is generated over F by the collection of all roots of all of the polynomials in the collection.)

Most books I have encountered define normal in a different way: one says that an algebraic extension $F \subseteq K$ is normal if whenever $f \in F[x]$ is an irreducible polynomial with a root in K, then f splits completely in $K[x]$.

Thus the point of exercise 13.4 #5 is to verify that the two definitions coincide, at least for extensions of finite degree.

I feel this exercise is tricky at this point, so here is a suggested outline of the more difficult direction, which is to prove that if K is a splitting field over F, then every irreducible polynomial in $F[x]$ with a root in F splits completely in $K[x]$. (If you want to try this problem without a hint, stop reading.)

Suggested outline: K is the splitting field over F for some single polynomial $f \in F[x]$ (since $[K : F] < \infty$). Suppose $g \in F[x]$ is an irreducible polynomial which has a root in K. Let $K \subseteq L$ where L is a splitting field over K for g. Then L is also a splitting field over F for fg. Show that if $\alpha_1, \alpha_2 \in L$ are any two roots of g, there there is an automorphism of L (i.e. isomorphism from L to itself) which is the identity map on F and which takes α_1 to α_2 (This is where you use theorems 8 and 27.) Prove also that any automorphism σ of L which is the identity map on F must send the elements of K into K (show that σ must permute the roots of f). Now put this all together.