Math 200b Winter 2011 Homework 5

Due 2/11/2011 by 5pm in homework box

All exercise numbers refer to Dummit and Foote, 3rd edition. I like to list all exercises that I think are interesting. It is good to think about extra exercises if you can (at least look over the statements of the ones you don't have time to do.) However, only the exercises that are marked with a star are to be handed in for grading.

Reading assignment: 14.1-14.2

Section 13.5: #7, 11*

Section 13.6: #3, 4*, 5*, 6, 9

Additional exercises not from the text (all to be handed in):

1. Prove that a field K is algebraically closed if and only if it has the following property: given any algebraic extension $K \subseteq L$, then $L = K$.

2. Let F be a field and let $F \subseteq K_1$, $F \subseteq K_2$ be two extensions each of which is an algebraic closure of F. Show that there is an isomorphism $\phi : K_1 \to K_2$ which is the identity map when restricted to F. This shows that an algebraic closure is unique up to isomorphism. (This is the final statement of Proposition 31 in the text, whose proof is omitted.)

(Hint: Consider the collection $\mathcal{S}$ of pairs (L, θ) such that $F \subseteq L \subseteq K_1$ and $\theta : L \to K_2$ is a homomorphism of fields which is the identity map when restricted to F. Make $\mathcal{S}$ into a poset where one pair (M, ρ) is defined to be greater than or equal to another (L, θ) if $L \subseteq M$, and ρ restricted to L is equal to θ. Use Zorn's lemma on this poset.)

3. An algebraic extension $F \subseteq K$ is called *purely inseparable* if for every $\beta \in K \setminus F$, the minimal polynomial of β over F is an inseparable polynomial.

(a). Suppose that $F \subseteq K$ is a purely inseparable extension where char $F = p$. Show that for any $\alpha \in K$, the minimal polynomial of α over F is of the form $f(x) = x^{p^n} - \alpha^{p^n}$ for some $n \geq 0$. In particular, this shows that $\alpha^{p^n} \in F$. (Hint: Proposition 38 in the text, which we did not cover in class, may be helpful.)

(b). Suppose that $F \subseteq K$ is an algebraic extension and that $\alpha \in K$ has a minimal polynomial over F which is inseparable. Show that $F \subseteq F(\alpha)$ is a purely inseparable extension.

As was realized only on the day this homework was due, this problem is incorrect as stated. Instead, can you find a counterexample to the statement of this problem?