Math 200b Winter 2011 Homework 6

Due 2/18/2011 by 5pm in homework box

All exercise numbers refer to Dummit and Foote, 3rd edition. I like to list all exercises that I think are interesting. It is good to think about extra exercises if you can (at least look over the statements of the ones you don't have time to do.) However, only the exercises that are marked with a star are to be handed in for grading.

Reading assignment: Continue to study 14.1-14.2, then begin reading 14.3-14.4. We will probably not cover the material in the first part of 14.4 on composite extensions.

Section 14.1: #1, 5, 7*, 9

Section 14.2: #1, 2, 4*, 5*, 8*, 11, 13*

Additional exercises not from the text (all to be handed in):

1. Let $F \subseteq E$ be a Galois extension and let K and L be intermediate fields, i.e. $F \subseteq K \subseteq E$ and $F \subseteq L \subseteq E$. Show that there is an isomorphism $\theta : K \to L$ which is the identity on F if and only if the subgroups of $G = \mathrm{Gal}(E/F)$ corresponding to K and L in the fundamental theorem are conjugate in G.

2. Let $p_1, \ldots, p_n$ be different prime numbers and let $E = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})$ as a subfield of $\mathbb{R}$. Show that $E/\mathbb{Q}$ is Galois and that $\mathrm{Gal}(E/\mathbb{Q})$ is elementary Abelian of order 2^n, i.e. isomorphic to a direct product of n copies of $\mathbb{Z}/2\mathbb{Z}$.
(Hint: Show that the fields $\mathbb{Q}(\sqrt{k})$ are all different as k runs over the $2^n - 1$ different products of distinct members of the set $\{p_1, \ldots, p_n\}$.)

3. This continues problem 2. Show in the situation of problem 2 that $E = \mathbb{Q}(\alpha)$, where $\alpha = \sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_n}$.
(Hint: determine how the 2^n elements of the Galois group G act on the elements $\sqrt{p_1}, \ldots, \sqrt{p_n}$. Then show that the orbit of α under G contains 2^n different elements.)