Math 200b Winter 2011 Homework 8

Due 3/4/2011 by 5pm in homework box

All exercise numbers refer to Dummit and Foote, 3rd edition. I like to list all exercises that I think are interesting. It is good to think about extra exercises if you can (at least look over the statements of the ones you don't have time to do.) However, only the exercises that are marked with a star are to be handed in for grading.

Reading assignment: You are not responsible for parts of the later sections of Chapter 14, some of which I consider to be TMI at the moment. You should read 14.5, excluding the last part about constructibility of the n-gon unless you are interested. We are covering only the fundamental theorem of algebra in 14.6, although we would talk more about the discriminant as well as norm and trace if we had more time. In 14.7, we are covering only the first part, not the solution of the cubic and the quartic. We are skipping 14.8. I will cover in more detail the topic of transcendence degree which is only sketched in 14.9.

Section 14.4: #3, 6*.

Section 14.5: #5, 7*, 10, 11, 12*

Additional exercises not from the text (all to be handed in):

1. Suppose that V is a nonzero vector space over an infinite field F. Show that V cannot be written as a union $V = W_1 \cup W_2 \cup \cdots \cup W_n$ where each W_i is a F-subspace of V with $W_i \subsetneq V$. Using this fact, prove that if a finite degree field extension $F \subseteq K$ (where F is infinite) has only finitely many intermediate fields, then $K = F(\alpha)$ for some $\alpha \in K$. This gives a different way of thinking about part of the proof of the theorem characterizing field extensions with a primitive element (Proposition 24 in the text.)

2. Let $F \subseteq K$ and suppose that L and E are intermediate fields such that E is Galois over F and $E \cap L = F$. If $\alpha \in L$ is algebraic over F and $f \in F[x]$ is the minimal polynomial over α over F, show that f is irreducible as an element of $E[x]$.

Give an example that shows the result above does not hold in general if E/F is finite degree but not Galois.

3. Let p be an odd prime, and let ζ be any fixed primitive pth root of unity in $\mathbb{C}$. Define the *Gauss sum*

$$\alpha = \sum_{i=0}^{p-1} \zeta^{i^2}.$$

Show that $E = \mathbb{Q}(\alpha)$ is the unique subfield of the cyclotomic field $\mathbb{Q}(\zeta)$ such that $[E : \mathbb{Q}] = 2$.

4. Suppose that $F \subseteq K$ is a Galois extension with $\mathrm{Gal}(K/F)$ abelian of order n. Assume that $x^n - 1$ already splits over F with distinct roots. Show that K is a splitting field over F of some polynomial of the form $(x^{n_1} - a_1)(x^{n_2} - a_2) \dots (x^{n_r} - a_r)$ where $a_i \in F$.

(Hint: Use the fundamental theorem of abelian groups)

5. let $f \in \mathbb{Q}[x]$ be irreducible and have prime degree p, where p is an odd prime. Assume that f has $p - 2$ real roots and 2 complex roots. Show that if K is the splitting field of f over $\mathbb{Q}$ then $\mathrm{Gal}(K/F) \cong S_p$, and so f is not solvable by radicals if $p \geq 5$.

(Hint: Think of the Galois group as a permutation group on the roots of f and find a p-cycle and a 2-cycle in the Galois group.)