# Math 200b Winter 2012 Final Exam

## March 19, 2012

Do as many of the problems as well as you can; the exam may be too long for you to finish. You may use major theorems proved in class or the textbook, but not if the whole point of the problem is reproduce the proof of a theorem proved in class or the textbook. Generally avoid quoting results proved only in homework exercises.

## NAME:

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| **Problem 1/ 10** | |
| **Problem 2 /10** | |
| **Problem 3 /10** | |
| **Problem 4 /10** | |
| **Problem 5/ 15** | |
| **Problem 6 /10** | |
| **Total /65** | |

1. (a)(5 pts). Reprove the result proved in class that if

$$0 \to M \to N \to P \to 0$$

is a short exact sequence of left $R$-modules and where $P$ is a projective $R$-module, then $N \cong M \oplus P$ as $R$-modules.

(b)(2 pts). Reprove the result proved in class that if $P$ is a projective $R$-module, then there is an $R$-module $Q$ such that $P \oplus Q$ is a free module.

(c)(3 pts). Give an example, with proof, of a ring $R$ and a projective $R$-module $P$ such that $P$ is not free.

2. (a)(5 pts). Carefully state the classification theorem of finitely generated modules over a PID (either invariant factor or elementary divisor form, your choice.) Explain why it implies that any torsionfree f.g. $R$-module, where $R$ is a PID, is free.

(b)(5 pts). Give examples, with proof, which show the following: (i) a f.g. torsionfree module over an integral domain $R$ need not be free; and (ii) an infinitely generated module over a PID $R$ need not be free.

3.(10 pts) Let $V$ and $W$ be finite dimensional vector spaces over a field $F$. Recall that a *simple* (or *pure*) tensor is an element of the tensor product $V \otimes_F W$ of the form $(v \otimes w)$ where $v \in V$, $w \in W$.

Determine necessary and sufficient conditions on $V$ and $W$ for the tensor product $T = V \otimes_F W$ to have the property that *every* element of $T$ is a simple tensor. Prove your answer.

4. Let $F \subseteq K \subseteq L$ be fields, where $[L : F] < \infty$. Suppose that $K/F$ and $L/K$ are Galois extensions.

(a)(5 pts). Suppose that for every automorphism $\sigma \in \mathrm{Gal}(K/F)$, $\sigma$ can be extended to an automorphism of $L$. Prove that $L/F$ is a Galois extension.

(b)(5 pts). Give an example, with proof, showing that $L/F$ need not be a Galois extension in general.

5. (a)(12 pts). Find the splitting field $K$ of $x^6 - 2$ over $\mathbb{Q}$, and show that $[K : \mathbb{Q}] = 12$. Show that $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to $D_{12}$, the dihedral group with 12 elements.

(b)(3 pts). How many distinct intermediate subfields $\mathbb{Q} \subseteq E \subseteq K$ are there with the property that $[K : E] = 2$ and $E/\mathbb{Q}$ is *not* Galois?

6. Let $F$ be a field with characteristic $p > 0$.

(a)(2 pts). Show that $F^p = \{a^p | a \in F\}$ is a subfield of $F$.

(b)(2 pts). Recall that $F$ is *perfect* if $F = F^p$. Give an explicit example, with justification, of a field $F$ which is not perfect.

(c)(6 pts). Prove that $F$ is perfect if and only if every irreducible polynomial in $F[x]$ is separable. (I think at least part of this was done in class, but you must reprove it.)