Math 200b Winter 2012 Homework 1

Due 1/20/2012 by 5pm in homework box (now in basement)

All exercise numbers refer to Dummit and Foote, 3rd edition.

Reading assignment: Read 10.1-10.3 and begin to read 12.1 (10.4 and 10.5 will be covered later). Skim 11.1, 11.2 to remind yourself about some facts from linear algebra (but skip the last part of 11.2 involving tensor products; we will not cover tensor products until later). We will assume some of the facts from these sections, so read these sections more closely if you feel rusty on linear algebra.

Assigned problems (all to be turned in)

I have stopped listing extra problems that are not to be handed in but might be good for practice. Generally, any number of the non-assigned problems near the beginning of each exercise section serve that purpose. Thinking about additional exercises is always good.

Section 10.1: **8, 9, 19, 21**

(In number 19 above, it is unspoken that V is being made into an $F[x]$-module where F acts by scalar multiplication and x acts as T.)

Section 10.2: **9**

Section 10.3: **5, 9, 10, 11**

(In number 9 above, the statement seems a little confusing. What they want you to prove is that M is irreducible if and only if *every* nonzero element of M generates M as a module.)

Additional problem:

In this problem you will verify the details of the correspondence between modules and maps to endomorphism rings which was stated but not proved in class. Each step is straightforward; the difficulty of this exercise lies in keeping careful track of definitions. First I restate the result and below I tell you which parts I want you to verify.

Fix a ring R and an abelian group M. Let S be the following set:

$$S = \{\text{All possible left } R\text{-modules with underlying abelian group } M\}.$$

You may also think of S as all possible ways of making the abelian group M into an R-module. Let T be the following set:

$$T = \{\text{All possible (unital) ring homorphisms } \phi : R \to \text{End}_{\mathbb{Z}}(M)\}.$$

Here, $\text{End}_{\mathbb{Z}}(M) = \text{Hom}_{\mathbb{Z}}(M, M)$ is the endomorphism ring, whose elements are all $\mathbb{Z}$-module homomorphisms (i.e. abelian group homomorphisms) from M to itself, with pointwise addition and product being composition of functions. You may assume this is a ring (if you didn't check this when you went over your class notes, think about it now.)

Given an element of S, that is, an R-module structure on M, we define an element of T as follows. For each $r \in R$, let $\phi_r : M \to M$ be given by $m \mapsto r \cdot m$, where $\cdot$ is the given R-module action on M. Then define $\phi : R \to \text{End}_{\mathbb{Z}}(M)$ by the formula $r \mapsto \phi_r$. This ϕ is an element of T. Altogether this produces a function $\Phi : S \to T$.

Given an element of T, that is, a ring homomorphism $\phi : R \to \text{End}_{\mathbb{Z}}(M)$, define an element of S as follows. We define an R-action on M by $r \cdot m = (\phi(r))(m)$. This makes M into an R-module and so altogether this produces a function $\Psi : T \to S$.

Then Φ and Ψ are bijective inverse functions so an R-module structure on M is essentially the same concept as a ring homomorphism from R to $\text{End}_{\mathbb{Z}}(M)$.

1. (a). Prove carefully that the definition of $\Phi : S \to T$ makes sense. To do this, show that for any $r \in R$ we have $\phi_r \in \text{End}_{\mathbb{Z}}(M)$, and that the formula $\phi : r \mapsto \phi_r$ is a unital ring homomorphism from R to $\text{End}_{\mathbb{Z}}(M)$.

(b). Prove carefully that the definition of $\Psi : T \to S$ makes sense. To do this, makes sure given a homomorphism $\phi : R \to \text{End}_{\mathbb{Z}}(M)$, the rule $r \cdot m = (\phi(r))(m)$ satisfies the module axioms.

(c). Show that $\Phi \circ \Psi : T \to T$ and $\Psi \circ \Phi : S \to S$ are the identity functions, so that Φ and Ψ are inverse bijections between the two sets.