Math 200b Winter 2012 Homework 4

Due 2/10/2012 by 5pm in homework box (now in basement)

All exercise numbers refer to Dummit and Foote, 3rd edition.

Reading assignment: 10.4, 10.5. The midterm exam will be on Monday February 13. It will cover the material up through tensor products (The first 4 homeworks.)

Assigned problems (all to be turned in)

Section 10.4: **3, 4, 11, 16**

Additional exercises not from the text (all to be handed in):

1. Let R be an integral domain and F its field of fractions. Show that if M is a torsion R-module, then $F \otimes_R M = 0$.

2. Let $F \subseteq K$ be a field extension; in other words, F and K are fields and F is a subring of K. Then K is an F-module by left multiplication, and so K is an F-algebra. Consider the ring $R = F[x]/(g(x))$ for some polynomial $g \in F[x]$. Then R is also naturally an F-vector space and in fact an F-algebra.

Now we saw in class that the tensor product of two F-algebras is again an F-algebra, so $K \otimes_F R$ is an F-algebra. Prove that this is isomorphic as an F-algebra to the ring $K[x]/(g(x))$. In other words, there is a ring isomorphism $K \otimes_F R \to K[x]/(g(x))$ which is also a F-vector space map.

3. $\mathbb{C}$ is an $\mathbb{R}$-algebra by left multiplication. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ as $\mathbb{R}$-algebras. (Hint: recall that $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$ and use problem 2.)

4. Let R be a commutative ring. Recall that a subset S of R is a *multiplicative system* if $1 \in S$ and S is closed under multiplication. As discussed in the fall, the localization of R at S is the ring of equivalence classes of formal fractions $RS^{-1} = \{r/s \,|\, r \in R, s \in S\}/\sim$, where $\sim$ is the equivalence relation $r_1/s_1 \sim r_2/s_2$ if and only if $t(s_2 r_1 - s_1 r_2) = 0$ for some $t \in S$, and where the ring operations in RS^{-1} are the usual ones for fractions.

One can also define a localization for modules. Let M be an R-module. The localized module MS^{-1} is defined to be the set of equivalence classes of formal fractions $MS^{-1} =$

$\{m/s \,|m \in M, s \in S\}/\sim$, where $\sim$ is the equivalence relation $m_1/s_1 \sim m_2/s_2$ if and only if $t(s_2 m_1 - s_1 m_2) = 0$ for some $t \in S$. Then MS^{-1} is naturally an RS^{-1}-module with action $(r/s) \cdot (m/t) = rm/st$. (You should check that this module action makes sense, but don't write that up.)

There is another useful construction of the localization of a module using the tensor product. The tensor product $RS^{-1} \otimes_R M$ is naturally a left RS^{-1}-module, where

$$(r/s) \cdot [(p/t) \otimes m] = (rp/st) \otimes m.$$

(This is really just the usual extension of scalars idea; we are making an R-module an RS^{-1}-module in the most natural way. Note though that we are extending scalars via the not necessarily injective ring homomorphism $\phi : R \to RS^{-1}$ given by $r \mapsto r/1$. See (6) on page 369 of the text.)

Now prove that $RS^{-1} \otimes_R M \cong MS^{-1}$ as RS^{-1}-modules. Thus localization of modules is just a special case of base extension.