MATH 200B WINTER 2021 FINAL

Instructions: As previously announced, this is an open book exam— you may use Isaacs, Dummit and Foote, and any class materials if you wish (lectures, homework writeups, course notes). You may not refer to any other textbooks or online sources.

You may make use of theorems proved in class, the notes, or one of the approved textbooks, but not if it trivializes the problem. Avoid quoting the results of homework exercises unless instructed that you can.

This is a 3 hour exam plus 10 minutes for downloading and 15 minutes for uploading. All problems are worth 15 points.

1. Let F be a field. Let $A \in M_n(F)$. Let $f = \mathrm{minpoly}_F(A) \in F[x]$.

(a). Let $\overline{F}$ be the algebraic closure of F. Show that $\mathrm{minpoly}_{\overline{F}}(A) = f$.

(b). Show that A is diagonalizable over $\overline{F}$ (that is, A is similar in $M_n(\overline{F})$ to a diagonal matrix) if and only if f is a separable polynomial.

(c). Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in M_3(F)$. Is A diagonalizable over $\overline{F}$? The answer may depend on the properties of F.

2. Let $F \subseteq K$ be a field extension with $[K : F] < \infty$. In this problem, if you find any results from homework problems helpful you can quote them here rather than redoing them. Note that a commutative ring R is called *reduced* if R has no nonzero nilpotent elements.

(a). Suppose that K/F is separable. Prove that the K-algebra $K \otimes_F K$ is reduced, but is not a domain unless $K = F$.

(b). Suppose that K/F is inseparable. Show that the K-algebra $K \otimes_F K$ is not reduced.

3. Let $f(x) = x^{12} - 3 \in \mathbb{Q}[x]$. Let K be the splitting field of f over $\mathbb{Q}$.

(a). Show that $G = \mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to D_{24}, a dihedral group of order 24. (Hint: a primitive 12th root of 1 is given by $e^{2\pi i/12} = \cos(\pi/6) + i\sin(\pi/6)$.)

(b). Let Z be the center of G. Let $E = \mathrm{Fix}(Z)$. Show that E is a splitting field over $\mathbb{Q}$ of some polynomial $g \in \mathbb{Q}[x]$. Find such a g.

Date: March 15, 2021.