Math 200b Winter 2021 Homework 2

Due 1/22/2020 by midnight on Gradescope

1. Let R be an integral domain which is noetherian (that is, such that every ideal of R is finitely generated as an ideal).

(a). Show that if every finitely generated torsionfree R-module is free, then R is a PID.

(b). Show that if every torsionfree R-module is free, then R is a field.

2. Let $R = F[x]$ where F is a field. Suppose that

$$M = R/(x^2 - 1) \oplus R/(x^2 + 1) \oplus R/(x^2 - 2x + 1) \oplus R/(x^3 + 1).$$

(a). Assuming that $F = \mathbb{Q}$ is the rational numbers, find the elementary divisors and invariant factors of M as an $F[x]$-module.

(b). Assuming that $F = \mathbb{C}$ is the complex numbers, find the elementary divisors and invariant factors of M as an $F[x]$-module.

3. Let M be a finitely generated torsion module over the PID R. Suppose that M has s elementary divisors and t invariant factors. Define

$$S = \{n \in \mathbb{N} | M \text{ is isomorphic to a direct sum of } n \text{ cyclic modules}\}.$$

(a) Show that $s = \max(S)$.

(b) Show that $t = \min(S)$.

4. In class we proved that a finitely generated module over a PID is the direct sum of its torsion submodule and a torsionfree (in fact, free) module. This exercise shows that the same is not true of infinitely generated modules in general.

Let $M = \prod_p \mathbb{Z}/p\mathbb{Z}$, where the product runs over all distinct (positive) prime numbers p. Consider M as an abelian group (i.e. $\mathbb{Z}$-module).

(a) Show that the torsion submodule of M is $\operatorname{Tors}(M) = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$.

(b) Show that $N = M/\operatorname{Tors}(M)$ is a *divisible* $\mathbb{Z}$-module. This means for any $x \in N$ and prime number p, there is $y \in N$ such that $py = x$.

(c) Show that M has no nonzero divisible submodules over $\mathbb{Z}$.

(d) Conclude that $M \not\cong \operatorname{Tors}(M) \oplus (M/\operatorname{Tors}(M))$.