Math 200b Winter 2021 Homework 4

Due 2/5/2020 by midnight on Gradescope

1. Let V be a vector space over a field F, with $v_1, v_2 \in V$ be linearly independent over F. Show that $v_1 \otimes v_1 + v_2 \otimes v_2 \in V \otimes_F V$ is an element which is not equal to any pure tensor $u \otimes w$ with $u, w \in V$.

2. Let R be a commutative ring.

(a). Show that for any ideals I and J of R, that

$$R/I \otimes_R R/J \cong R/(I + J)$$

as R-modules.

(b). Suppose that R is a PID and M and N are finitely generated torsion R-modules. Show that $M \otimes_R N$ is also finitely generated as an R-module. Describe the elementary divisors of $M \otimes_R N$ in terms of the elementary divisors of M and N.

3. Let $F \subseteq K$ be an inclusion of fields. Then K is an F-algebra in a natural way. Since K and $F[x]$ are F-algebras, $K \otimes_F F[x]$ is also an F-algebra.

(a). Prove that $K \otimes_F F[x]$ is actually a K-algebra and that there is a K-algebra isomorphism $K \otimes F[x] \cong K[x]$.

(b). Consider the ring $R = F[x]/(g(x))$ for some $g \in F[x]$. Prove that $K \otimes_F R \cong K[x]/(g(x))$ as K-algebras.

(c). Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ as $\mathbb{C}$-algebras. (Hint: $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$.)

4. Let R be an integral domain with field of fractions F. Let M be a finitely generated R-module. Show that $F \otimes_R M$ is a finite-dimensional F-vector space. We define the *rank* of M to be $\text{rk}(M) = \dim_F(F \otimes_R M)$.

(a). If M is a torsion R-module, then $F \otimes_R M = 0$.

(b). Show that if $\text{rk}(M) = r$, then r is equal to largest natural number n such that M contains a submodule N which is free of rank n.

(c). If R is a PID, then $\text{rk}(M)$ is equal to the rank of M as we defined it earlier, as the unique number r such that $M \cong R^r \oplus T$ where T is a torsion module.