Math 200b Winter 2021 Homework 5

Due 2/19/2020 by midnight on Gradescope

1. Let M be a finitely generated R-module over a PID R. Prove that if M is flat, then M is free.

2. Show that the following polynomials are irreducible:
(a). $2x^3 + 2x + 5$ in $\mathbb{Q}[x]$.
(b). $x^2 + y^2 - 1$ as an element of the ring $\mathbb{Q}[x, y]$.

3. Show that the irreducible polynomials in $\mathbb{R}[x]$ all have degree 1 or 2.
(Hint: compare factorization of a polynomial in $\mathbb{R}[x]$ over $\mathbb{R}$ and over $\mathbb{C}$. You may assume that any polynomial splits over $\mathbb{C}$.)

4. Let $F \subseteq K$ be a field extension with $[K : F] = 2$. If $a \in K$, by $\sqrt{a}$ we mean an element of K whose square is a.
(a) Assume that F does not have characteristic 2. Show that $K = F(\sqrt{a})$ for some $a \in F$. (Hint: think about completing the square).
(b). Show that the result of part (a) fails in general if F has characteristic 2.

5. Let R be a commutative domain which is an algebra over the field F, such that $\dim_F R = n < \infty$. Prove that R is a field. (Hint: for $0 \neq a \in R$, consider the map $R \to R$ given by left multiplication by a.)