Math 200b Winter 2021 Homework 6

Due 2/26/2020 by midnight on Gradescope

1. Let F be a field of characteristic not 2. Let $F \subseteq K$ be a field extension, and let $a, b \in F$ be elements, neither of which is a square in F. Let $\sqrt{a}, \sqrt{b} \in K$ be roots of the polynomials $x^2 - a, x^2 - b \in F[x]$, respectively. Prove that $[F(\sqrt{a}, \sqrt{b}) : F] = 4$ if and only if ab is not a square in F, and $[F(\sqrt{a}, \sqrt{b}) : F] = 2$ otherwise.

2(a). Let $F \subseteq K$ be a field extension. Suppose that $F \subseteq K_1 \subseteq K$ and $F \subseteq K_2 \subseteq K$ where K_1 and K_2 are subfields of K. The composite field K_1K_2 is defined to be the smallest subfield of K containing both K_1 and K_2. Show that if $[K_1 : F] < \infty$ and $[K_2 : F] < \infty$ then K_1K_2 can also be described as the usual notation for products of subsets of a ring suggests:

$$K_1K_2 = \left\{ \sum_{i=1}^{d} a_i b_i \middle| a_i \in K_1, b_i \in K_2, d \geq 0 \right\} \subseteq K.$$

(Hint: show that K_1K_2 is a subring of K— then why is it forced to be a subfield?)

(b). Show that if $[K_1 : F] < \infty$ and $[K_2 : F] < \infty$ as in (b), then there is a surjective homomorphism of F-algebras $\theta : K_1 \otimes_F K_2 \to K_1K_2$ given by $\theta(a \otimes b) = ab$. Prove that θ is an isomorphism if and only if $[K_1K_2 : F] = [K_1 : F][K_2 : F]$.

(c). Show that the $\mathbb{Q}$-algebra $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3})$ is a field which is isomorphic to $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

3. Suppose that $f \in F[x]$ has degree n and that K is a splitting field for f over F. Show that $[K : F] \leq n!$.

4. Let K be the splitting field of $x^6 - 4$ over $\mathbb{Q}$. Find $[K : \mathbb{Q}]$.

5. Let $\alpha = \sqrt{2 + \sqrt{2}} \in \mathbb{C}$ and let f be the minimal polynomial of α over $\mathbb{Q}$.
(a). Compute f.
(b). Let K be a splitting field for f over $\mathbb{Q}$. Find $[K : \mathbb{Q}]$.

6. Let K be a field with $\operatorname{char} K = p > 0$. Define $K^p = \{a^p | a \in K\}$ and recall that K is called *perfect* if $K^p = K$. In this problem let K be a nonperfect field.

(a). Consider the extension $F = K^p \subseteq K$. Show that for every $\alpha \in K - F$, $\text{minpoly}_F(\alpha)$ is an inseparable polynomial of degree p.

(b). Show that if $[K : F] > p$ then the extension K/F has no primitive element, i.e. there is no $\gamma \in K$ such that $K = F(\gamma)$.

(c). Give an explicit example where $[K : F] < \infty$ and the situation of part (b) occurs; so finite degree extensions in characteristic p do not always have primitive elements.