Math 200b Winter 2021 Homework 7

Due 3/5/2020 by midnight on Gradescope

1. In this problem you show that $G = \mathrm{Aut}(\mathbb{R}) = 1$.

(a). Show that every element of $\mathrm{Aut}(\mathbb{R})$ fixes $\mathbb{Q}$ pointwise; that is $\mathrm{Aut}(\mathbb{R}) = \mathrm{Gal}(\mathbb{R}/\mathbb{Q})$.

(b). Let $\sigma \in G$. Prove that σ takes squares to squares and hence takes the set of positive numbers to itself. Using this conclude that $a < b$ implies $\sigma(a) < \sigma(b)$.

(c). Prove that σ is a continuous function.

(d). Prove that any continuous function $\mathbb{R} \to \mathbb{R}$ which fixes every element of $\mathbb{Q}$ is the identity, and so $\sigma = 1$.

(Remark: By constrast, it is possible to show that $\mathrm{Aut}(\mathbb{C})$ is uncountable!)

2. Let $F \subseteq K$ be a finite degree field extension. Suppose that $F \subseteq E_1 \subseteq K$ and $F \subseteq E_2 \subseteq K$ are intermediate fields and that E_1/F and E_2/F are normal extensions.

(a). Show that $(E_1 \cap E_2)/F$ is a normal extension.

(b). Show that $(E_1E_2)/F$ is a normal extension. The composite E_1E_2 is defined on the previous homework.

3. Let $F \subseteq K$ be a finite degree Galois extension and let $F \subseteq E \subseteq K$ and $F \subseteq L \subseteq K$ be intermediate fields. Show that there is an isomorphism of fields $\theta : E \to L$ with $\theta|_F = 1_F$ if and only if the subgroups $\mathrm{Gal}(K/L)$ and $\mathrm{Gal}(K/E)$ of $G = \mathrm{Gal}(K/F)$ are conjugate subgroups in G.

4. Let F be a field of characteristic p. Recall that we proved that if $f \in F[x]$ is inseparable and irreducible, then $f = \sum_{i=0}^{n} b_i x^{ip}$ for some $b_i \in F$. We can also write this condition as $f = g(x^p)$ where $g = \sum_{i=0}^{n} b_i x^i \in F[x]$.

(a). Prove that any irreducible polynomial $f \in F[x]$ is of the form $g(x^{p^k})$ for some irreducible, separable polynomial $g \in F[x]$ and some $k \geq 0$.

(b). An algebraic extension $F \subseteq K$ is called *purely inseparable* if for all $\alpha \in K - F$, the minimal polynomial of α over F is inseparable. Prove that $F \subseteq K$ is purely inseparable if and only if every $\alpha \in K$ satisfies $\alpha^{p^k} \in F$ for some $k \geq 0$.

5. Let F be a field of characteristic $p > 0$, and let $\mathbb{F}_p$ be its prime subfield. Let K be the splitting field of F of the polynomial $f(x) = x^p - x - a \in F[x]$. Let $\alpha \in K$ be a root of f and assume that $\alpha \notin F$. Show that

(a). $\alpha + i$ is also a root of f, for all $i \in \mathbb{F}_p$.

(b). $K = F(\alpha)$.

(c). f is separable and irreducible in $F[x]$, and K/F is Galois. (hint: all roots of f have minimal polynomials over F of the same degree).

(d). For each $i \in \mathbb{F}_p$ there is an automorphism $\sigma_i \in G = \mathrm{Gal}(K/F)$ such that $\sigma_i(\alpha) = \alpha + i$. Moreover $G = \{\sigma_i | i \in \mathbb{F}_p\}$ and G is cyclic of order p.

6. Let $p_1, p_2, \ldots, p_n$ be different prime numbers and let $E = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})$ as a subfield of $\mathbb{C}$.

(a). Show that $E/\mathbb{Q}$ is Galois and that $\mathrm{Gal}(E/\mathbb{Q})$ is Elementary abelian of order 2^n. (Hint: show that the fields $\mathbb{Q}(\sqrt{k})$ are all different as k runs over the $2^n - 1$ different products of nonempty subsets of the set $\{p_1, p_2, \ldots, p_n\}$.)

(b). Show that $E = \mathbb{Q}(\alpha)$, where $\alpha = \sqrt{p_1} + \cdots + \sqrt{p_n}$. (Hint: determine how the 2^n elements of the Galois group G act on the elements $\sqrt{p_1}, \ldots, \sqrt{p_n}$. Then show that the orbit of α under G contains 2^n different elements.)