Math 200c Spring 2011 Homework 1

Due 4/8/2011 by 5pm in homework box

All exercise numbers refer to Dummit and Foote, 3rd edition. I like to list all exercises that I think are interesting. It is good to think about extra exercises if you can (it is good to look over the statements of the ones you don't have time to do.) However, only the exercises that are marked with a star are to be handed in for grading.

Reading assignment: Read 10.4-10.5. We are covering most of 10.4 and some of 10.5. This homework set is only about tensor products (10.4).

Section 10.4: 4, 7, 11, 12*, 15, 16*, 17*, 18, 19, 20, 21*, 23, 25*, 26

Additional exercises not from the text (all to be handed in):

1. Let $F \subseteq K$, $F \subseteq L$ be two field extensions of K.

(a). Suppose that both K/F and L/F are simple extensions of finite degree, so we can write $K \cong F[x]/(f(x))$ and $L \cong F[x]/(g(x))$ for some irreducible polynomials $f, g \in F[x]$.

Prove that $K \otimes_F L$ is isomorphic to the ring $K[x]/(g(x))$.

(b). With the same hypotheses as in (a), show that $K \otimes_F L$ is a field if and only if g is irreducible over K.

In case $K \otimes_F L$ is a field, choose any extension M of K which contains a root of f and a root of g (for example, a splitting field of $f(x)g(x)$ over F or even an algebraic closure of F.) Then M contains subfields K' and L' such that K' is F-isomorphic to K and L' is F-isomorphic to L. Show that $K \otimes_F L$ is isomorphic to the composite of K' and L' inside M.

(c). Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the ring $\mathbb{C} \times \mathbb{C}$ (use part (a)).

2. Let R be an integral domain. Recall that a subset S of R is a *multiplicative system* if $1 \in S$ and S is closed under multiplication. As discussed in the fall, the localization of R at S is the ring $RS^{-1} = \{r/s \,|r \in R, s \in S\}/\sim$ where $\sim$ is the equivalence relation $r_1/s_1 \sim r_2/s_2$ if and only if $s_2 r_1 = s_1 r_2$, and the ring operations in RS^{-1} are the usual ones for fractions. Notice that we have an inclusion of rings $R \subseteq RS^{-1}$.

Let M be an R-module. The localized module MS^{-1} is defined to be the set $MS^{-1} = \{m/s \,|m \in M, s \in S\}/\sim$ where $\sim$ is the equivalence relation $m_1/s_1 \sim m_2/s_2$ if and only if

$s_2m_1 = s_1m_2$. MS^{-1} is naturally an RS^{-1}-module with action $(r/s) \cdot (m/t) = rm/st$. (You should check that this module action makes sense, but don't write that up.)

Prove that MS^{-1} is isomorphic as an RS^{-1}-module to the base extension $RS^{-1} \otimes_R M$. Thus localization of modules is just a special case of base extension.

3. Let R be an integral domain. Let K be the field of fractions of R, so $R \subseteq K$.

(a). Suppose that P is a torsion R-module. Show that $K \otimes_R P = 0$.

(b). Show that if $N \subseteq M$ is a submodule such that $N \cong R^m$ as R-modules and M/N is torsion, then $K \otimes_R M \cong K^m$ as K-modules. Conclude that $\dim_K(K \otimes_R M)$ is equal to the *rank* of M as we defined it last quarter. (Hint: use exercise 12.1 #2(a) which you did last quarter to see that m is the rank of M.)