Math 200c Spring 2011 Homework 3

Due Friday 5/6/2011 by 5pm in homework box

All exercise numbers refer to Dummit and Foote, 3rd edition. I list all exercises that seem interesting. Think about extra exercises if you can (it is good to at least look over the statements of the ones you don't have time to do.) Only the exercises that are marked with a star are to be handed in for grading.

Reading: Finish reading 15.1-15.3. I will give a handout from the book "Noncommutative algebra" by Farb and Dennis that contains the material we will cover on noncommutative rings. We will cover (parts of) chapters 1 and 2 of the handout (Chapter 0 of the handout is background which we mostly already covered.)

Section 15.2: 2, 3, 5*, 6, 8, 10, 11*, 12, 26*, 27.

Section 15.3: 5, 10*.

Additional exercises not from the text (all to be handed in):

1. Recall the notion of Krull dimension for a commmutative ring, which was introduced in Math 200a: A chain of prime ideals in a ring R of the form $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ is said to have *length* n, and the Krull dimension of R, $\dim R$, is the supremum of the lengths of all chains of prime ideals of R.

If $R \subseteq S$ is an integral extension of rings, it is a basic fact that (i) $\dim R = \dim S$ (see Theorem 26 in the text, and Exercise 5.2 #17.) Another fundamental fact is that (ii) a polynomial ring $k[x_1, \dots, x_n]$ over a field k has Krull dimension $\dim k[x_1, \dots, x_n] = n$.

Assume facts (i) and (ii) above. Now prove the following proposition, which gives a useful alternative characterization of the Krull dimension.

Proposition 0.1 *Let R be a commutative f.g. k-algebra which is a domain, and let F be the field of fractions of R. Since R is a k-algebra, R contains a copy of k and so we have an associated field extension $k \subseteq F$. Prove that* $\dim R$ *is equal to the transcendence degree of F over k.*

(Hint: use Noether normalization.)

2. Let $k = \mathbb{C}$ and let $J = (xy - 1, x^2 + y) \subseteq k[x, y]$. Find $Z(J) \subseteq \mathbb{A}^2$, and show how it decomposes as a union of irreducible components. Then find $\mathcal{I}(Z(J))$, writing it as an intersection of prime ideals.

3. We used Noether normalization in the proof of the weak Nullstellensatz. Really the only thing we needed Noether normalization for could be separated out as the following lemma:

Lemma 0.2 *Let F be a field which is also a f.g. k-algebra. Then F/k is algebraic, and moreover $[F : k] < \infty$.*

In this problem, you study a different proof of the lemma above that avoids Noether normalization, but which depends on k being an uncountable field; so this proof is not quite as general. Still, this is a useful method to see because one is often interested in the case $k = \mathbb{C}$, where such cardinality techniques are available.

(a). Assume that k is uncountable, and let $k(x)$ be the field of rational functions in one variable over k. Thinking of $k(x)$ as a vector space over k, prove that the set of elements $\{(x-a)^{-1} \mid a \in k\}$ is linearly independent over k. Conclude that $\dim_k k(x)$ is an uncountable cardinal number.

(b). Let k be any field. Prove that if R is a f.g. commutative k-algebra, then $\dim_k R$ is a countable cardinal number.

(c). Prove Lemma 0.2 above in the special case that k is uncountable, using (a) and (b).

4. Consider $R = \mathbb{C}[x, y]/(x^i - y^j)$ for some $i, j \geq 2$ with $\gcd(i, j) = 1$.

(a). Show that R is isomorphic to the subring $\mathbb{C}[t^i, t^j]$ of a polynomial ring $\mathbb{C}[t]$.

(b). Show that the integral closure of $\mathbb{C}[t^i, t^j]$ in $\mathbb{C}(t)$ is $\mathbb{C}[t]$.

(c). Show that R is not integrally closed in its field of fractions.

(FYI: the fact that R is not integrally closed in its field of fractions is an algebraic reflection of the geometric fact that the curve $Z(x^i - y^j) \subseteq \mathbb{A}^2$ has a singular point at the origin.)