Problem 1. (15 pts)

Let $G$ be a group with 117 elements which contains an element of order exactly 9. Classify all such groups $G$ up to isomorphism.

Problem 2. (10 pts)

Let $G$ be a finite group and let $p$ be the smallest prime dividing $|G|$. Assume that $G$ has a unique subgroup $H$ of order $p$. Show that $H$ is contained in the center of $G$.

*Hint:* For each $g \in G$, prove that the permutation $\sigma_g(h) = ghg^{-1}$ of the set $H \setminus \{e\}$ is trivial by investigating its order.

Problem 3. (10 pts)

Consider a commutative ring $A$ with unity and let $n$ be its nilradical. Show that the following statements are equivalent:

(i) $A$ has only one prime ideal
(ii) every element in $A$ is either a unit or nilpotent
(iii) $A/n$ is a field.

Problem 4. (15 pts)

Let $K$ denote a splitting field of $x^7 - 1$ over $\mathbb{Q}$ inside $\mathbb{C}$. Determine all subfields of $K$. Express each subfield in the form $\mathbb{Q}(\alpha)$ for some $\alpha \in \mathbb{C}$, where you must justify that $\alpha$ is a primitive element for the subfield if it is not obvious.

Problem 5. (15 pts)

Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $K = k(t)$ be a purely transcendental extension in one indeterminate $t$. Let $n \geq 1$ be any integer, and let $L$ be the splitting field of the polynomial $x^n - t$ over $K$. It may be helpful in this problem to write $n = p^i m$ where $\gcd(m, p) = 1$.

(i) (7 pts) Show that $L = K(\alpha)$ where $\alpha$ is any root of $x^n - t$ in $L$.
(ii) (8 pts) Let $G = \text{Aut}(L/K)$ be the group of all automorphisms of $L$ fixing $K$ pointwise, and let $F = \text{Fix}(G)$ be the subfield of $L$ of elements fixed by $G$. Calculate $[F : K]$. 
Problem 6. (15 pts)

Given vector spaces $V$ and $W$ over the complex numbers, suppose that $\phi : V \rightarrow V$ and $\psi : W \rightarrow W$ are $\mathbb{C}$-linear transformations.

(i) (6 pts) Show that there is a unique linear transformation

$$\phi \otimes \psi : V \otimes_\mathbb{C} W \rightarrow V \otimes_\mathbb{C} W$$

with the property that

$$(\phi \otimes \psi)(v \otimes w) = \phi(v) \otimes \psi(w)$$

for all $v \in V$, $w \in W$.

(ii) (9 pts) Let $V$ and $W$ be finite-dimensional of complex dimensions $m$ and $n$ respectively. Prove that

$$\det(\phi \otimes \psi) = \det(\phi)^n \det(\psi)^m.$$ 

*Hint: Choose $\mathbb{C}$-bases for $V$ and $W$ such that the matrices representing $\phi$ and $\psi$ have a special form.*

Problem 7. (10 pts)

Let $R$ be an integral domain. Prove or give an example to disprove (with justification):

(i) (3 pts) If $M$ is a torsion $R$-module, then $\text{Ann}_R(M) \neq 0$.

(ii) (3 pts) If $M$ is a free $R$-module, then $M$ is torsionfree.

(iii) (4 pts) If $M$ is a torsionfree $R$-module, then $M$ is free.

Problem 8. (15 pts)

Assume $A$ is a commutative ring with unity, and let $f_1, \ldots, f_n \in A$ generate the unit ideal $(1)$. Assume that the rings of fractions $A_{f_1}, \ldots, A_{f_n}$ are Noetherian. Prove that $A$ is Noetherian.