Math 200c Spring 2012 Homework 1

Due 4/13/2012 by 5pm in homework box (now in basement)

Reading assignment: Read Chapters 1-3 of Atiyah-Macdonald. We have covered a good portion of the material in these chapters previously. Since I will not assume everyone has immediate access to a copy of Atiyah-Macdonald, I will type out all exercises below even if they come from Atiyah-Macdonald. All rings R are commutative with 1.

Assigned problems (all to be turned in)

1. For an ideal I of a ring R, let $\sqrt{I} = \{x \in R | x^n \in I \text{ for some } n > 0\}$ be the radical of I. We proved that $\sqrt{I}$ is the intersection of all prime ideals which contain I. Prove the following facts for any ideals I and J:

(a) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

(b) $\sqrt{I} = R$ if and only if $I = R$.

(c) $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.

(d) If I is prime then $\sqrt{I^n} = I$ for any $n > 0$.

2. Recall that in a ring R, the Zariski topology on the set of prime ideals $X = \text{Spec } R$ is the topology whose closed sets are the sets $V(I) = \{P \in \text{Spec } R | P \supseteq I\}$, as I ranges over all ideals. Prove that these sets really are the closed subsets of a topological space, i.e. prove that $\emptyset, X$ are among these closed sets, that an arbitrary intersection of closed sets is closed, and that a finite union of closed sets is closed.

3. Let R be a ring, and put the Zariski topology on $X = \text{Spec } R$. For any $f \in R$, define $X_f = V(f)^c = X \setminus V(f)$, i.e. the complement in X of the closed set $V(f)$; so the sets X_f are open. They are called *principal open subsets* of X. Prove the following statements.

(a) The sets $\{X_f | f \in R\}$ form a basis for the Zariski topology. In other words, every open set in X is a union of such principal open subsets.

(b) $X_f \cap X_g = X_{fg}$.

(c) $X_f = \emptyset$ if and only if f is nilpotent.

(d) $X_f = X$ if and only if f is a unit.

(e) $X_f = X_g$ if and only if the ideals (f) and (g) have the same radical.

(f) X is quasi-compact (this means that every open covering of X has a finite subcover. This property is generally just called compact in topology). (Hint: show it is enough to consider a cover by a family of principal open sets X_{f_α}. Show that the f_α generate the unit ideal, and so finitely many of them generated the unit ideal.)

4. In a ring R, let Σ be the set of all ideals in which every element is a zerodivisor (recall that x is a zerodivisor if there is $y \neq 0$ such that $xy = 0$.) Show that the set Σ has maximal elements and that every maximal element is a prime ideal. Conclude that the set of zerodivisors of R is a union of prime ideals. (Remark: we will do better later and show that in a noetherian ring R the set of zerodivisors is a union of finitely many primes called the *associated primes* of the ring.)

5. Let $\mathbb{A}^n = \{(a_1, \ldots, a_n) | a_i \in \mathbb{C}\}$ be affine n-space over $\mathbb{C}$. Recall that a subset of $\mathbb{A}^n$ is closed if it is of the form $V(Y) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n | f(a_1, \ldots, a_n) = 0 \text{ for all } f \in Y\}$ for some subset Y of the ring $\mathbb{C}[x_1, \ldots, x_n]$. Show that $\mathbb{A}^n$ has DCC on closed subsets, i.e. that if $X_1 \supseteq X_2 \supseteq X_3 \ldots$ for some closed subsets X_i of $\mathbb{A}^n$, then there is N such that $X_n = X_N$ for all $n \geq N$.

6. Let $I = (xy, (x-y)z) \subseteq \mathbb{C}[x, y, z]$. Prove that $\sqrt{I} = (xy, xz, yz)$. Calculate explicitly the zero set $V(I) = V(\sqrt{I}) \subseteq \mathbb{A}^3$. Prove that $\sqrt{I}$ is not a prime ideal, and write $\sqrt{I}$ explicitly as an intersection of prime ideals.