Math 200c Spring 2012 Homework 2

Due 4/29/2012 by 5pm in homework box (now in basement)

Reading assignment: Read Chapters 3-4 of Atiyah-Macdonald. Since I will not assume everyone owns a copy of Atiyah-Macdonald, I will type out all exercises below even if they come from Atiyah-Macdonald. All rings R are commutative with 1. However, I do recommend you get some kind of access to Atiyah-Macdonald so you can read along.

Assigned problems (all to be turned in)

1. *Note that this problem was revised on April 18th to add the hypothesis that N is finitely generated.* Let R be a local ring with maximal ideal $\mathfrak{m}$. Let M and N be R-modules, where N is a finitely generated R-module. Show that a homomorphism of R-modules $f : M \to N$ is surjective if and only if the induced homomorphism $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective.

2. *Note that this problem was revised on April 18th to add a missing hypothesis from part (a) and give an extensive hint.*

(a). Let R be a ring and M, N R-modules. Let S be any multiplicative system in R. There is a natural map

$$\phi : S^{-1}\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N),$$

of $S^{-1}R$-modules, defined in the obvious way (if $f : M \to N$, and $s \in S$, then $\phi(f/s)$ is the homomorphism sending m/t to $f(m)/st$.)

A module M is called *finitely presented* if there is a right exact sequence

$$R^m \xrightarrow{f} R^n \xrightarrow{g} M \to 0,$$

for some finite $m, n \geq 0$ (in other words, $\mathrm{Im}(f) = \ker(g)$ and g is surjective.) Prove that if M is finitely presented, then the map ϕ above is an isomorphism.

(*Hint:* First prove directly that ϕ is an isomorphism when M is free of finite rank, i.e. isomorphic to some R^m. Then use the short exact sequence $R^m \to R^n \to M \to 0$ to produce a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}\operatorname{Hom}_R(M, N) \arrow[r] \arrow[d, "\phi"] & S^{-1}\operatorname{Hom}_R(R^n, N) \arrow[r] \arrow[d] & S^{-1}\operatorname{Hom}_R(R^m, N) \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \arrow[r] & \operatorname{Hom}_{S^{-1}R}(S^{-1}R^n, S^{-1}N) \arrow[r] & \operatorname{Hom}_{S^{-1}R}(S^{-1}R^m, S^{-1}N)
\end{tikzcd}$$

in which the rows are left exact and the last two verticals are isomorphisms; a diagram chase shows the first vertical is an isomorphism also.)

(b). Prove that being a finitely presented projective module is a local property. Namely, let M be a finitely presented R-module. Then show that M is a projective module if and only if M_P is a projective R_P-module for all prime ideals P of R. (Hint: use the characterization of projective which says that M is projective if and only if for all surjective R-module maps $f : A \to B$, the induced map $f_* : \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B)$ is surjective.)

3. Let X be a topological space. Any subset Y of X is then a topological space with the *subspace topology* if we declare the closed sets in Y to be exactly those sets of the form $(Y \cap Z)$ where Z is closed in X. Recall also that a map $f : X \to W$ of topological spaces is *continuous* if for every closed subset Z of W, $f^{-1}(Z)$ is closed in X. (You probably have met this definition using open sets instead, but that it is equivalent). A *homeomorphism* is a continuous map which is bijective and such that the inverse map is also continuous.

(a). Let $\phi : R \to T$ be a homomorphism of rings. For a prime ideal P of T, define $\phi^*(P) = \phi^{-1}(P)$. In this way, we define a map $\phi^* : \mathrm{Spec}\ T \to \mathrm{Spec}\ R$. Prove that this is a continuous map (where both spaces have the Zariski topology, of course.)

(b). Let S be a multiplicative system and let $\phi : R \to RS^{-1}$ be the natural map. By part (a) we have a continuous map $\phi^* : \mathrm{Spec}\ S^{-1}R \to \mathrm{Spec}\ R$. Show that ϕ^* is injective, and that ϕ^* gives a homeomorphism between $\mathrm{Spec}\ S^{-1}R$ and the image of ϕ^* (with its subspace topology inherited from Spec R).

(c). Suppose S is the set of all elements not in a prime ideal P. Prove that the image of $\phi^* : \mathrm{Spec}\ S^{-1}R \to \mathrm{Spec}\ R$ is equal to the intersection of all open sets in $\mathrm{Spec}\ R$ which contain P. (This underlies the intuition that geometricially $\mathrm{Spec}\ S^{-1}R$ is telling us about "local" behavior near the point P.)

4. In the polynomial ring $K[x, y, z]$ where K is a field, let $P_1 = (x, y)$, $P_2 = (x, z)$, and let $I = P_1P_2$. Find a primary decomposition of I, and find the set of primes associated to I. Which of the primes associated to I are minimal and which are embedded? Recalling that only the primary components associated to minimal primes need be uniquely determined, find a different primary decomposition of I.

5. Let R be a ring and a S a multiplicative system in R. Consider the localization map $R \to S^{-1}R$. For any ideal I of R, the ideal I^{ec} is called the *saturation* of I (with respect to S.) Recall that we proved that explicitly $I^{ec} = \{r \in R | sr \in I \text{ for some } s \in S\}$.

Now let P be a prime in R. Taking S to be the set of all elements of R not in P, The *nth symbolic power of* P, $P^{(n)}$, is defined to be the the saturation of P^n with respect to S, namely

$$P^{(n)} = \{x \in R | sx \in P^n \text{ for some } s \notin P\}.$$

(a). Show that $P^{(n)}$ is a P-primary ideal.

(b). If the ideal P^n has a primary decomposition, prove that $P^{(n)}$ must be its P-primary component.

(c). Show that $P^{(n)} = P^n$ if and only if P^n is P-primary.