Math 200c Spring 2015 Homework 4

Due Friday, June 5 by 3pm

Reading assignment: Read Chapters 7-9 of Atiyah-Macdonald. I will count 3 out of 4 homeworks this quarter, so you should definitely do this one if you did not do one of the earlier ones, while you could safely omit this one if you are happy with your scores on homeworks 1-3.

Assigned problems:

1. Let A be a valuation ring which is not a field. Let $v : K \setminus \{0\} \to G$ be the associated valuation, where K is the field of fractions of A and G is a totally ordered Abelian group, as in Exercises 30-31 of Chapter 5.

(a). Prove that the principal ideals of A are exactly the ideals $I_g = \{a \in A | v(a) \geq g\}$ as g ranges over elements of G with $g \geq 0$.

(b). Prove that A is a *Bezout domain*: given $x, y \in A$, then $(x, y) = (z)$ for some z. As a consequence, every finitely generated ideal of A is principal.

(c). Prove that if A is a noetherian ring, then A is a discrete valuation ring, that is, $G = \mathbb{Z}$.

2. We used the Artin-Tate lemma to prove the following result which is the key lemma in the proof of Hilbert's Nullstellensatz:

Lemma 0.1 *Let F be a field which is also a f.g. k-algebra. Then F/k is algebraic, and moreover $[F : k] < \infty$.*

In this problem, you study an interesting different proof of the lemma which depends on k being an uncountable field; so this proof is not quite as general. Still, this is a useful method to see because one is often interested in the case $k = \mathbb{C}$, where such cardinality techniques are available.

(a). Assume that k is uncountable, and let $k(x)$ be the field of rational functions in one variable over k. Thinking of $k(x)$ as a vector space over k, prove that the set of elements $\{(x-a)^{-1} \mid a \in k\}$ is linearly independent over k. Conclude that $\dim_k k(x)$ is an uncountable cardinal number.

(b). Let k be any field. Prove that if R is a f.g. commutative k-algebra, then $\dim_k R$ is a countable cardinal number.

(c). Prove Lemma 0.1 above in the special case that k is uncountable, using (a) and (b).

3. Let A be a finitely generated k-algebra for some field k. Prove that the following are equivalent:

(i) $\dim_k A < \infty$.
(ii) A is an artinian ring.
(Hint: for (ii) implies (i), first reduce to the local case. If A is artinian local with max ideal $\mathfrak{m}$, apply the weak Nullstellensatz to the residue field $A/\mathfrak{m}$ and go from there.)

4. Let A be a Dedekind domain, and S a multiplicative system in A.

(a). Show that $S^{-1}A$ is either again a Dedekind domain, or else is the entire field of fractions of A.

(b). Show that extension of ideals induces a surjective homomorphism from the ideal class group of A to the ideal class group of $S^{-1}A$.

5. Let A be a Dedekind domain. Prove that every ideal I of A is generated by at most 2 elements.
(Hint: For any nonzero $a \in A$, consider the ring $A/(a)$; one needs to show that every ideal in this ring is principal. We have $(a) = P_1^{i_1} \dots P_n^{i_n}$ for some maximal ideals P_j and $i_j \geq 1$. Show that as rings, $A/(a) \cong \prod_j A/P_j^{i_j}$. Prove that for any max ideal P, the ring A/P^i is isomorphic to the localization $(A/P^i)_P = A_P/(P_P)^i$. Since A_P is a dvr, conclude that every ideal of A/P^i is principal. Then deduce that every ideal of $A/(a)$ is principal.)