Math 201a Fall 2023 Exercises Set 1

October 6, 2023

These are completely optional! Happy to hear from you if you have questions or comments about these (in particular in case you think any of them has an error).

1. Recall that $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category of all functors $\mathcal{C} \to \mathcal{D}$, where for $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the morphisms $\mathrm{Hom}_{\mathrm{Fun}}(F, G)$ are all natural transformations $\eta : F \to G$.

Show that $\eta \in \mathrm{Hom}_{\mathrm{Fun}}(F, G)$ is an isomorphism in the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ if and only if η is a natural isomorphism.

2. Prove the result stated in class that a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if F is full, faithful, and essentially surjective.

(Hint/sketch: For each each $N \in \mathcal{D}$, there is $M \in \mathcal{C}$ such that $F(M) \cong N$. Choose arbitrarily such an object M and fix a specific isomorphism $i_N : F(M) \to N$ in $\mathcal{D}$. Define $G(N) = M$; given $g \in \mathrm{Hom}_{\mathcal{D}}(N_1, N_2)$, use faithfulness and fullness of F to see that there is a unique morphism $G(g) \in \mathrm{Hom}_{\mathcal{C}}(G(N_1), G(N_2))$ such that $F(G(g)) = i_{N_2}^{-1} \circ g \circ i_{N_1}$. Show that this defines a functor $G : \mathcal{D} \to \mathcal{C}$, then prove that G is the required quasi-inverse to F.)

4. Recall that for $M \in \mathcal{C}$, $h_M : \mathcal{C} \to \mathbf{Set}$ is the functor given by $h_M(N) = \mathrm{Hom}_{\mathcal{C}}(M, N)$, with action on a morphism $f : N_1 \to N_2$ given by $h_M(f) : \mathrm{Hom}_{\mathcal{C}}(M, N_1) \to \mathrm{Hom}_{\mathcal{C}}(M, N_2)$ where $[h_M(f)](g) = f \circ g$. A functor $F : \mathcal{C} \to \mathbf{Set}$ is called *representable* if there is a natural isomorphism $F \cong h_M$ for some M. Yoneda's Lemma is the following result:

Lemma 0.1 *For any $M \in \mathcal{C}$ and functor $F \in \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$, there is a bijection*

$$\mathrm{Hom}_{\mathrm{Fun}}(h_M, F) \longleftrightarrow F(M).$$

(a) Prove Yoneda's Lemma. (Hint: given $\eta \in \mathrm{Hom}_{\mathrm{Fun}}(h_M, F)$, η_M is a morphism $\eta_M : h_M(M) = \mathrm{Hom}_{\mathcal{C}}(M, M) \to F(M)$ and so $\eta_M(1_M) \in F(M)$. This defines the map in one direction.)

(b) Recall that there is a functor $H : \mathcal{C}^{op} \to \text{Fun}(\mathcal{C}, \mathbf{Set})$ given by $H(M) = h_M$. Show that H is full and faithful. Conclude that $\mathcal{C}^{op}$ is equivalent to the full subcategory of $\text{Fun}(\mathcal{C}, \mathbf{Set})$ whose objects are all representible functors.

(c) Show that for any category $\mathcal{C}$, there is a non-representable functor $\mathcal{C} \to \mathbf{Set}$.

5. Consider the category k-Vec for a field k. In class we defined the dualization functor $D : k\text{-}\mathbf{Vec}^{op} \to k\text{-}\mathbf{Vec}$, where $D(V) = V^* = \text{Hom}_k(V, k)$. Then id and D^2 are both functors $k\text{-}\mathbf{Vec} \to k\text{-}\mathbf{Vec}$. Check all of the details of the result given in class that there is a natural transformation $\eta : \text{id} \to D^2$ of functors $k\text{-}\mathbf{Vec} \to k\text{-}\mathbf{Vec}$ for a field k. Verify carefully that when restricted to the category k-**vec** of finite-dimensional vector spaces, $\eta : \text{id} \to D^2$ is a natural isomorphism.

6. Prove the following equivalences of categories which are mentioned in class:

1. $k\text{-}\mathbf{Rep}_G \simeq kG\text{-}\mathbf{Mod}$ for any field k and group G, where kG is the group algebra.
2. $(R \otimes_{\mathbb{Z}} R^{op})\text{-}\mathbf{Mod} \simeq R\text{-}\mathbf{Bimod}$ for any ring R.

7. Recall that a bifunctor is a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, where $\mathcal{C} \times \mathcal{D}$ is a product of categories $\mathcal{C}$ and $\mathcal{D}$. Given such a bifunctor F, for every fixed $M \in \mathcal{C}$ we get a functor $F_M : \mathcal{D} \to \mathcal{E}$ where $F_M(N) = F(M, N)$ for $N \in \mathcal{D}$ and for $g \in \text{Hom}_{\mathcal{D}}(N_1, N_2)$ we have $F_M(g) = F(1_M, g)$.

Using this idea, show there is an equivalence of categories

$$\text{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \text{Fun}(\mathcal{C}, \text{Fun}(\mathcal{D}, \mathcal{E})).$$

8. Prove the following results mentioned in class:

(a) If $\mathcal{C}$ is a category in which the product $M \prod N$ of any two objects M, N exists, and $\mathcal{C}$ has a terminal object T, then $(\mathcal{C}, \prod, a, T, l, r)$ is a monoidal category. In particular, there is a natural sense in which $\prod$ also acts on morphisms and so is a bifunctor. Here $a_{M,N,P}$ is the canonical isomorphism. Given $M \in \mathcal{C}$, $l_M = p_2 : T \prod M \to M$ and $r_M = p_1 : M \prod T \to M$ are the projection maps. (Use the fact that T is terminal and the universal property of the product to show l_M and r_M are isomorphisms).

(b) Dually, if $\mathcal{C}$ is a category in which the coproduct $M \coprod N$ of any two objects M, N exists, and $\mathcal{C}$ has an initial object I, then $(\mathcal{C}, \coprod, a, I, l, r)$ is a monoidal category for the appropriate choice of l, r.

(c) Write down these monoidal categories explicitly, by finding what the products, coproducts, initial and terminal objects are, when $\mathcal{C}$ is **Rng**, **Top**, R-**Mod**, **Cat**, or other common categories.

9. Let M be a monoid (that is, a set M with a binary associative operation $\cdot$ and a unit element $1 \in M$ such that $x \cdot 1 = x = 1 \cdot x$ for all $x \in M$). Monoids form a category **Mon** with morphisms being defined in the obvious way.

To a monoid M we can associate a category $\mathcal{C}(M)$ with exactly one object X, where $\mathrm{Hom}_{\mathcal{C}(M)}(X, X) = M$, with composition of morphisms being given by multiplication in M.

(a) Show that there is an equivalence of categories $\mathbf{Mon} \simeq \mathbf{One\text{-}object\text{-}Cat}$, where the latter category is the full subcategory of **Cat** consisting of categories that have exactly one object.

(b) If G is a group, then in particular it is a monoid so we get the one-object category $\mathcal{C}(G)$. Show that there is an equivalence of categories $k\text{-}\mathbf{Rep}_G \simeq \mathrm{Fun}(\mathcal{C}(G), k\text{-}\mathbf{Vec})$.

(c) Let $\mathcal{C} = \mathcal{C}(M)$ for a monoid M. We can try to make this into a monoidal category $(\mathcal{C}, \otimes, a, X, l, r)$ where $X \otimes X = X$ (the only possible choice) and where given $x, y \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, the action of the bifunctor on morphisms is $x \otimes y = x \cdot y : X \to X$. Let a, l, r be the obvious identifications. For what monoids M is this a monoidal category?