Math 201a Fall 2023 Exercises Set 3

November 8, 2023

1. Let $\mathcal{C}$ be a preabelian category. We defined $\mathcal{C}$ to be abelian if, for any morphism $f : M \to N$ in $\mathcal{C}$, the induced morphism $\widehat{f} : \mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.

Show that a preabelian category is abelian if and only if every monomorphism is a kernel of some morphism and every epimorphism is a cokernel of some morphism. This gives an alternative way to define an abelian category.

2. Let Ab be the category of Abelian groups, which is an abelian category. Find full subcategories of Ab which are

(a) preadditive but not additive;

(b) additive but not preabelian;

(c) preabelian but not abelian.

(Hint: (c) is the trickiest. One possible example is the subcategory DivAb of divisible groups. This category has kernels and cokernels but the kernel in DivAb does not coincide with the kernel in the larger category Ab.)

3. Let $\mathcal{C}$ be any (small) category and $\mathcal{D}$ an abelian category. Let $\mathcal{F} = \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the category of functors from $\mathcal{C}$ to $\mathcal{D}$ with morphisms the natural transformations. Show that is naturally an abelian category again, using the structure of $\mathcal{D}$. For instance, if $\eta : F \to G$ is a morphism of functors in this category, then the kernel of η is the functor K such that $K(M) = \ker F(M) \to G(M)$ for each M.

4. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories.

(a) Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{D}$ be additive functors where (F, G) is an adjoint pair. Show that F is right exact and G is left exact. (Hint: To show G is left exact, show that if $0 \to M \to N \to P$ is a left exact sequence then $0 \to \mathrm{Hom}(-, GM) \to \mathrm{Hom}(-, GN) \to \mathrm{Hom}(-, GP)$ is left exact in the functor category $\mathrm{Fun}(\mathcal{C}, \mathrm{Ab})$, which is abelian by problem 3. Then apply Yoneda's lemma.

(b) Let $(\mathcal{C}, \otimes, a, \mathbb{1}, l, r)$ is a monoidal category, where $\mathcal{C}$ is abelian, and $\otimes$ is bilinear on morphisms. Suppose that this category is rigid, that is, every object has a left and right dual. Show that the monoidal product $\otimes$ is automatically biexact, that is, for any object $M \in \mathcal{C}$ the functors $M \otimes -$ and $- \otimes M$ are exact.

5. Let k be a field and let $A = M_n(k)$ be the k-algebra of $n \times n$ matrices under matrix multiplication. Show that if $n > 1$, there is no coalgebra structure (A, Δ, ϵ) on A which makes A into a bialgebra.

6. Let V be a vector space over a field k, and let V^* indicate the k-linear dual, that is $V^* = \mathrm{Hom}_k(V, k)$.

1. Consider the linear transformation
$$i : V^* \otimes_k V^* \to (V \otimes_k V)^*$$
where for $f, g \in V^*$, $[i(f \otimes g)](v \otimes w) = f(v)g(w)$. Show that i is always injective, and is surjective if and only if $\dim_k V < \infty$.

2. Using the map in (1), show that for any coalgebra (C, Δ, ϵ) over k (regardless of whether it is finite dimensional) there is a natural dual algebra structure on C^*.

7. When (A, m, u) is a k-algebra with $\dim_k A = \infty$, the fact that $i : A^* \otimes A^* \to (A \otimes_k A)^*$ is not an isomorphism prevents A^* from having a dual coalgebra structure in general. However there is something one can do. Define
$$A^\circ = \{f \in A^* | f(I) = 0 \text{ for some ideal } I \text{ with } \dim_k(A/I) < \infty\}.$$
This is called the *finite dual* of A; in words, it consists of those functionals that kill an ideal of finite codimension. Show that A° is naturally a coalgebra with $\Delta = m^*$, $\epsilon = u^*$, suitably interpreted.

(Remark: while this construction allows for a reasonable theory of dual coalgebras (and together with problem 7, bialgebras and Hopf algebras) in the infinite-dimensional case, it doesn't always lead to interesting results. If an algebra A has no ideals of finite codimension, then $A^\circ = 0$.)

8. Show that if $(B, m, u, \Delta, \epsilon)$ is a bialgebra, then the finite dual B° is again a bialgebra, and similarly if H is a Hopf algebra then so is H°. Give an explicit description of this dual Hopf algebra H° when $H = k[x]$ with $\Delta(x) = 1 \otimes x + x \otimes 1$ and $\epsilon(x) = 0$ (this is the universal enveloping algebra of a 1-dimensional Lie algebra).

9. Let $(H, m, u, \Delta, \epsilon, S)$ be a Hopf algebra over k. Verify the following results stated in class:

(a) S is an antihomomorphism of algebras, i.e. $S(ab) = S(b)S(a)$ for $a, b \in H$.

(b) S is an antihomomorphism of coalgebras, i.e. $\Delta(a) = \mu_{12} \circ \Delta(a)$ for $a \in H$, where $\mu_{12} : H \otimes_k H \to H \otimes_k H$ switches the tensor coordinates.

(Hint: we observed previously that a tensor product of coalgebras is a coalgebra, so $H \otimes_k H$ is a coalgebra. Consider the $\mathrm{Hom}_k(H \otimes_k H, H)$ as an algebra under the convolution product as defined in class. Show that both of the maps $f_1 : a \otimes b \mapsto S(ab)$ and $f_2 : a \otimes b \mapsto S(b)S(a)$ are inverses to $m : H \otimes_k H \to H$ under convolution, and so $f_1 = f_2$. Part (2) can be proved in a dual manner).

10. Let $(H, m, u, \Delta, \epsilon, S)$ be a Hopf algebra over k, such that S is bijective and so S^{-1} makes sense. Let $(H\text{-mod}, \otimes_k, a, k, l, r)$ be the monoidal category of finite-dimensional H-modules.

We checked some of the details that this category has left duals but not the right duals. Verify the claim made in class that any $V \in H\text{-mod}$ has a right dual ${}^*V$ defined as the usual vector space dual $\mathrm{Hom}_k(V, k)$ with $[h \cdot f](v) = f(S^{-1}(h)v)$ for $h \in H$, $f \in V^*$, $v \in V$. (Note that by problem 8, S is an anti-homomorphism of algebras, so the same is true of S^{-1}.)