1. Suppose that $Q$ is any quiver. The \textit{opposite quiver} of $Q$ is the quiver $Q^{\text{op}}$ with the same vertex set as $Q$, and one arrow $\alpha^* : j \to i$ for each arrow $\alpha : i \to j$ in $Q$. In other words, $Q^{\text{op}}$ is formed from $Q$ by switching the direction of all of the arrows.

Given a left module $M$ over $KQ$, show that there is a corresponding representation of $Q^{\text{op}}$ given by putting $V_i = \epsilon_i M$ and $\phi_{\alpha^*} : V_j \to V_i$ given by left multiplication by $\alpha \in KQ$. Show that an analogous proof as the one we used for right modules gives an equivalence of categories between the category $KQ$-$\text{Mod}$ of left $KQ$-modules and the category $\text{Rep}_{K}(Q^{\text{op}})$ of representations of $Q^{\text{op}}$.

2. Fix any quiver $Q$ and a field $K$. Let $\mathcal{C} = \text{rep}_{K} Q$ be the category of \textit{finite-dimensional} representations of $Q$ (note the lowercase $r$ on rep) and let $\mathcal{D} = \text{rep}_{K} Q^{\text{op}}$ be the category of finite-dimensional representations of $Q^{\text{op}}$.

(a). Show that there is a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ which does the following. Given a rep $(V, \phi)$ of $Q$, $F(V, \phi)$ is the rep $(V^*, \phi^*)$ of $Q^{\text{op}}$ which has $(V^*)_i = (V_i)^* = \text{Hom}_K(V_i, K)$ and for $\alpha : i \to j$ in $Q$, has $\phi_{\alpha^*} = (\phi_{\alpha})^* : V_j^* \to V_i^*$; that is, $(\phi_{\alpha})^*(\psi) = \psi \circ \phi_{\alpha}$ for $\psi : V_j \to K$. The action of $F$ on morphisms is the obvious one; I leave it to you to define it.

Prove that $F$ is a duality (that is, there is a contravariant functor $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to identity functors on $\mathcal{D}$ and $\mathcal{C}$ respectively.)

(b). We proved in class that there is an equivalence of categories $H : \text{rep}_{K}(Q) \to \text{fd-}KQ$ where $\text{fd-}KQ$ means the category of finite-dimensional right $KQ$-modules. We also proved there is a duality $D : \text{fd-}KQ \to KQ$-$\text{fd}$, where $KQ$-$\text{fd}$ is the category of finite-dimensional left $KQ$-modules. By problem 1, there is an equivalence of categories $J :$
$KQ$-fd $\rightarrow \text{rep}_K(Q^{op})$. Show that $J \circ D \circ H$ is naturally isomorphic to the duality $F$ described in part (a).

3. Consider the acyclic quiver $Q$ given by

\[
\begin{array}{cccccccc}
1 & & & & & & & \\
\downarrow & & & & & & & \\
3 & & 2 & & 4 & & 5 & & 6 \\
\end{array}
\]

Find explicitly all simple, indecomposable projective, and indecomposable injective representations of this quiver.

4. Verify the details of how Hom sets get bimodule structures, as follows:

(a). If $M$ is a $(B,A)$-bimodule and $N$ is an $(C,A)$-bimodule, show that $\text{Hom}_A(M, N)$ is a $(C,B)$-bimodule where $[c\phi](m) = c(\phi(m))$ and $[\phi b](m) = \phi(bm)$.

(b). If $M$ is an $(A,B)$-bimodule and $N$ is an $(A,C)$-bimodule, show that $\text{Hom}_A(M, N)$ is a $(B,C)$-bimodule where $[b\phi](m) = \phi(mb)$ and $[\phi c](m) = \phi(m)c$.

5. Let $Q$ be the Kronecker quiver

\[
\begin{array}{ccc}
1 & \alpha & 2 \\
\downarrow \beta & & \\
2 & & \\
\end{array}
\]

Let $A = KQ$. Let $(V, \phi)$ be the (infinite-dimensional) representation of $Q$ given by $V_1 = V_2 = K[t]$ (the polynomial ring in one variable) with $\phi_\alpha$ the identity map and $\phi_\beta$ multiplication by $t$. Let $M$ be the right $KQ$-module corresponding to the rep $(V, \phi)$.

Show that $M$ is indecomposable, but that $\text{End} \, M$ is not local. Thus endomorphism rings of indecomposable modules which are not of finite length need not be local, even over a finite-dimensional algebra.
6. Let $Q$ be an acylic quiver with vertices $1, \ldots, n$.

(a). Let $P$ be a finite-dimensional projective right module over $KQ$, and let $(V, \phi)$ be the corresponding representation. Show that for all arrows $\alpha$ in $Q$, the map $\phi_\alpha$ in the representation is an injective vector space map. (Hint: reduce to the indecomposable case).

(b). Similarly, if $E$ is a finite-dimensional injective right module over $KQ$, show that the maps in the corresponding rep $(V, \phi)$ are all surjective.