Notations and Conventions

• $K$ is a field.
• $V$ and $W$ are $K$-vector spaces.
• Unadorned $\otimes$ symbols are always over $K$.

Homework 1

1.1 Problem 1

Proposition 1.1. The natural map $\psi_{V,W}: V^* \otimes W^* \to (V \otimes W)^*$ is always injective, and it is surjective if and only if one of $V$ or $W$ is finite-dimensional.

The first statement may be proven using bases, and this proof is reasonable. However, I prefer going through one of my favorite lemmas about the tensor product.

Definition 1.2. $\mathcal{S} \subseteq V^*$ is called separating if $f(v) = 0$ for every $f \in \mathcal{S}$ implies $v = 0$.

For two linear functionals $f \in V^*$, $g \in W^*$, abuse notation slightly by writing $f \otimes g$ also for $\psi_{V,W}(f \otimes g) \in (V \otimes W)^*$.

Lemma 1.3. Let $\mathcal{S} \subseteq V^*$, $\mathcal{T} \subseteq W^*$ be separating sets. If $x \in V \otimes W$ is such that $(f \otimes g)(x) = 0$, for every $f \in \mathcal{S}$ and $g \in \mathcal{T}$, then $x = 0$.

Proof. Write $x = \sum_{j=1}^{m} v_j \otimes w_j$, where $v_1, \ldots, v_m \in V$ and $w_1, \ldots, w_m \in W$, where we take $w_1, \ldots, w_m \in W$ linearly independent. (This is possible basically because finite-dimensional vector spaces have bases.) Now, the condition implies that if $f \in \mathcal{S}$, then

$$0 = (f \otimes g)(x) = \sum_{j=1}^{m} f(v_j)g(w_j) = g\left(\sum_{j=1}^{m} f(v_j)w_j\right),$$

for every $g \in \mathcal{T}$. Since $\mathcal{T}$ is separating, $\sum_{j=1}^{m} f(v_j)w_j = 0$. Since $w_1, \ldots, w_m \in W$ are linearly independent, it follows that $f(v_1) = \cdots = f(v_m) = 0$, for every $f \in \mathcal{S}$. Since $\mathcal{S}$ is separating, we get $v_1 = \cdots = v_m = 0$. Therefore, $x = 0$, as desired. □
The key observation is that if \( e_V : V \hookrightarrow V^{**} \) is the natural map, then \( e_V(V) \subseteq V^{**} \) is a separating set in \((V^*)^*\). Indeed, if \( f \in V^* \), then \( e_V(v)(f) = f(v) \), for all \( v \in V \).

**Proof of Injectivity Part of Proposition 1.1.** Suppose \( x \in V^* \otimes W^* \) is such that \( \psi_{V,W}(x) \equiv 0 \). For every \( v \in V \) and \( w \in W \), note the functional \( e_V(v) \otimes e_W(w) : V^* \otimes W^* \to K \) satisfies
\[
(e_V(v) \otimes e_W(w))(x) = \psi_{V,W}(x)(v \otimes w) = 0.
\]
By Lemma 1.3 and the observation above, we conclude \( x = 0 \), as desired.

For the second part of Proposition 1.1, we make an observation.

**Lemma 1.4.** Let \( (v_n)_{n \in \mathbb{N}} \in V^\mathbb{N} \), \( (w_n)_{n \in \mathbb{N}} \in W^\mathbb{N} \) be sequences in \( V \) and \( W \), respectively. If \( h := \sum_{j=1}^m f_j \otimes g_j \in \text{im} \psi_{V,W} \subseteq (V \otimes W)^* \), then
\[
\text{rank } \begin{bmatrix} h(v_1 \otimes w_1) & \cdots & h(v_1 \otimes w_n) \\ \vdots & \ddots & \vdots \\ h(v_n \otimes w_1) & \cdots & h(v_n \otimes w_n) \end{bmatrix} \leq m,
\]
for all \( n \in \mathbb{N} \).

**Proof.** Note that for all \( j, k \in [n] \), we have
\[
h(v_j \otimes w_k) = \sum_{\ell=1}^m f_\ell(v_j) g_\ell(w_k) = (F_n G_n)_{jk},
\]
where
\[
F_n = \begin{bmatrix} f_1(v_1) & \cdots & f_m(v_1) \\ \vdots & \ddots & \vdots \\ f_1(v_n) & \cdots & f_m(v_n) \end{bmatrix} \in K^{n \times m} \quad \text{and} \quad G_n = \begin{bmatrix} g_1(w_1) & \cdots & g_1(w_n) \\ \vdots & \ddots & \vdots \\ g_m(w_1) & \cdots & g_m(w_n) \end{bmatrix} \in K^{m \times n},
\]
whence it follows that
\[
\text{rank } \begin{bmatrix} h(v_1 \otimes w_1) & \cdots & h(v_1 \otimes w_n) \\ \vdots & \ddots & \vdots \\ h(v_n \otimes w_1) & \cdots & h(v_n \otimes w_n) \end{bmatrix} = \text{rank}(F_n G_n) \leq \text{rank } F_n \leq m,
\]
as desired.

We are now ready for the rest of Proposition 1.1.

**Proof of Second Part of Proposition 1.1.** We first show \( \psi_{V,W} \) is not surjective if \( V \) and \( W \) are both infinite-dimensional. In this case, there are sequence \((v_n)_{n \in \mathbb{N}} \in V^\mathbb{N}\), \((w_n)_{n \in \mathbb{N}} \in W^\mathbb{N}\) of linearly independent vectors in both \( V \) and \( W \). Then we know \((v_n \otimes w_n)_{n \in \mathbb{N}} \) is linearly independent in \( V \otimes W \). Completing it to a basis (or choosing a complementary subspace), we conclude there exists \( \varphi \in (V \otimes W)^* \) such that
\[
\varphi(v_j \otimes w_k) = \delta_{jk},
\]
for all \( j, k \in \mathbb{N} \). But then \([\varphi(v_j \otimes w_k)]_{j,k\in[n]} = I_n \in K^{n \times n}\), for all \( n \in \mathbb{N} \). Since \( \text{rank } I_n = n \), we conclude from Lemma 1.4 that \( \varphi \not\in \text{im } \psi_{V,W} \), and therefore \( \psi_{V,W} \) is not surjective.
Now, in the case that (without loss of generality) $W$ is finite-dimensional, let $n := \dim W$. Since all maps are natural, we may as well assume $W = K^n$. In this case, we can consider the identifications

$$(V^*)^n \cong (V^* \otimes K)^n$$

$$\cong (V^* \otimes K^*)^n$$

$$\cong V^* \otimes (K^*)^n$$

$$\cong V^* \otimes (K^n)^*.$$

and

$$(V \otimes K^n)^* \cong ((V \otimes K)^n)^*$$

$$\cong (V^n)^* \cong (V^*)^n$$

(These all work out because direct sums, i.e., coproducts, and direct products, i.e., products, of finitely many vector spaces are the same.) One may check easily that under these identifications,

$$(V^*)^n \cong V^* \otimes (K^n)^* \xrightarrow{\psi_{V,K^n}} (V \otimes K^n)^* \cong (V^*)^n$$

is the identity map, so that $\psi_{V,K^n}$ is surjective.

\[\square\]

1.2 Problem 2

**Proposition 1.5.** Let $I \subseteq V^*$ and $J \subseteq W^*$ be subspaces. Then $(I \otimes J)^\perp = I^\perp \otimes W + V \otimes J^\perp$.

As above, there is a proof of this fact using bases, but I prefer to go through Lemma 1.3.

Let $V_1, V_2, W_1, W_2$ all be $K$-vector spaces.

**Lemma 1.6 (Flatness).** If $T_1: V_1 \to W_1$ and $T_2: V_2 \to W_2$ are injective linear maps, then $T_1 \otimes T_2: V_1 \otimes V_2 \to W_1 \otimes W_2$ is injective as well.

**Proof.** We first observe $\mathcal{S} := \{f \circ T_1 \in V_1^*: f \in W_1^*\}$ and $\mathcal{T} := \{g \circ T_2 \in V_2^*: g \in W_2^*\}$ are separating sets. Indeed, if $v \in V_1$ is such that $f(T_1v) = 0$, for all $f \in W_1^*$, then $T_1v = 0$ (because $W_1^*$ is a separating set). Since $T_1$ is injective, we conclude $v = 0$, as desired. The same argument works for $\mathcal{T}$.

Now, suppose $x \in V_1 \otimes V_2$ is such that $(T_1 \otimes T_2)(x) = 0$. Then, for all $f \in W_1^*$ and $g \in W_2^*$, we have

$$((f \circ T_1) \otimes (g \circ T_2))(x) = ((f \otimes g) \circ (T_1 \otimes T_2))(x) = (f \otimes g)(0) = 0.$$  

By the previous paragraph and Lemma 1.3, we conclude $x = 0$, as desired. \[\square\]

**Lemma 1.7 (Quotients).** If $V_1 \subseteq V$ and $W_1 \subseteq W$ are subspaces and $\pi_1: V \to V/V_1$ and $\pi_2: W \to W/W_1$ are the natural quotient maps, then $\ker(\pi_1 \otimes \pi_2) = V_1 \otimes W + V \otimes W_1$.

**Proof.** We did this in class. \[\square\]
Theorem 1.8. If $T_1: V_1 \to W_1$ and $T_2: V_2 \to W_2$ are linear, then

$$\ker(T_1 \otimes T_2) = \ker T_1 \otimes V_2 + V_1 \otimes \ker T_2.$$ 

Proof. Restricting the codomains of $T_1$ and $T_2$ to get $\tilde{T}_1: V_1 \to \text{im } T_1$ and $\tilde{T}_2: V_2 \to \text{im } T_2$, we get $T_1 \otimes T_2$ as the composition

$$V_1 \otimes V_2 \xrightarrow{T_1 \otimes T_2} \text{im } T_1 \otimes \text{im } T_2 \xrightarrow{\iota_1 \otimes \iota_2} W_1 \otimes W_2,$$

where $\iota_j: \text{im } T_j \hookrightarrow W_j$ is inclusion, for $j \in \{1, 2\}$. Since $\iota_1 \otimes \iota_2$ is injective by Lemma 1.6, 

$$\ker(T_1 \otimes T_2) = \ker(\tilde{T}_1 \otimes \tilde{T}_2) = \ker \tilde{T}_1 \otimes V_2 + V_1 \otimes \ker \tilde{T}_2 = \ker T_1 \otimes V_2 + V_1 \otimes \ker T_2,$$

by Lemma 1.7 and the First Isomorphism Theorem. 

Now, the key observation is that if $I \subseteq V^*$ is a subspace and $e_I^V := \rho_I^V \circ e_V: V \to I^*$ is the natural map $e_V: V \hookrightarrow V^{**}$ followed by the restriction map $\rho_I^V: (V^*)^* \to I^*$ defined by $f \mapsto f|_I$, then

$$I^\perp = \ker e_I^V,$$

by definition of $\perp$. 

Proof of Proposition 1.5 Consider the map $e_I^V \otimes e_J^W: V \otimes W \to I^* \otimes J^*$. By Theorem 1.8 we have that

$$\ker(e_I^V \otimes e_J^W) = \ker e_I^V \otimes W + V \otimes \ker e_J^W = I^\perp \otimes W + V \otimes J^\perp.$$ 

But also, the universal property of the tensor product implies that $e_{V \otimes W}^{I \otimes J}$ is the composition

$$V \otimes W \xrightarrow{e_I^V \otimes e_J^W} I^* \otimes J^* \xrightarrow{\psi_{I,J}} (I \otimes J)^*$$

where $\psi_{I,J}$ is the injective map from Proposition 1.1. We conclude that

$$(I \otimes J)^\perp = \ker(e_{V \otimes W}^{I \otimes J}) = \ker(e_I^V \otimes e_J^W) = I^\perp \otimes W + V \otimes J^\perp,$$

as claimed. 

□
Math 207A HW 1 Problem 3

April 17, 2020

1. Since $C$ is a coalgebra, and $c \in C$ is group-like we have

\[ (1 \otimes \epsilon) \circ \Delta(c) = c \]  
\[ (1 \otimes \epsilon)(c \otimes c) = c \]  
\[ c \cdot \epsilon(c) = c \]  
\[ \epsilon(c) = 1 \]  

(Note $c \neq 0$ since 0 is not a group-like element.)

2. Suppose the group-like elements are not linearly independent, then there is a linear depending relation

\[ \alpha_1 c_1 + \cdots + \alpha_n c_n = 0, \]

where $c_i, c$ are distinct group-like elements in $C$ and $\alpha_i$ are elements in $k$ for $i \in \{1, \cdots, n\}$. Assume $\{c_1, \cdots, c_n\}$ are the smallest possible set that has linear dependency, then we observe that $\alpha_i \neq 0$, $\forall i \in \{1, \cdots, n\}$ (Otherwise we would have a smaller linear dependent set). Furthermore, if $n = 1$ then we have

\[ \alpha_1 c_1 = 0 \]

which is a contradiction since 0 is not a group-like element in $C$, so $n > 1$.

Then we can write

\[ c_n = \sum_{i=1}^{n-1} \beta_i c_i, \]

where $\beta_i \in k/\{0\}$ for all $i \in \{1, \cdots, n-1\}$. Since this is a smaller set of $c_i$'s, it has to be linear independent.

Now we apply $\Delta$ to both sides of the equation above and we get

\[ c_n \otimes c_n = \sum_{i=1}^{n-1} \beta_i c_i \otimes c_i. \]

Thus we get $1 = n - 1$ by looking at the rank of both sides, so we get $\beta_1 c_1 + \beta_2 c_2 = 0 \Rightarrow c_2 = \beta_1 c_1$. By applying $\epsilon$ to both sides we get $\epsilon(c_2) = \epsilon(\beta_1 c_1) = 1$, so $\beta_1 = 1 \Rightarrow c_1 = c_2$, which reaches a contradiction.
4. (a) Let $D \subseteq C$ be a subcoalgebra of a grouplike coalgebra. Let $\{g_i\}$ be a basis of grouplike elements in $C$. Let $d \in D$. Then $d = \sum_i a_i g_i$ for some $a_i \in k$. Then $\Delta(d) = \sum_i a_i g_i \otimes g_i$, but also $\Delta(d) \in D \otimes D \subseteq D \otimes C$ so $\Delta(d) = \sum_i v_i \otimes g_i$ for some $v_i \in D$. Since the $g_i$ are linearly independent, we have that $a_i g_i = v_i \in D$ for all $i$. Therefore each $g_i \in D$ when $a_i \neq 0$. So $d$ is in the span of grouplike elements. Hence all of $D$ is the span of grouplike elements. Since the grouplike elements are linearly independent, $D$ is grouplike.

(b) If $C, D$ are grouplike coalgebras, then they have bases of grouplike elements $\{c_i\}$ and $\{d_j\}$ respectively.

A basis for $C \oplus D$ is given by $\{(c_i, 0), (0, d_j)\}$, and note that the coproduct on $C \oplus D$ is given by first applying $\Delta_C \otimes \Delta_D$ to $C \oplus D$ and then distributing the direct sum canonically. In other words, $(c, 0) \mapsto (c \otimes c, 0 \otimes 0) \mapsto (c, 0) \otimes (c, 0)$ and similarly for $(0, d)$ where $c \in C$ and $d \in D$ is grouplike. So the $(c_i, 0)$ and the $(0, d_j)$ are grouplike.

For the tensor product, $\{c_i \otimes d_j\}$ is a basis, and the coproduct is taken by first applying $\Delta_C \otimes \Delta_D$ to $C \otimes D$ and then applying $\tau_{23}$. That is, $(c, d) \mapsto (c \otimes c) \otimes (d \otimes d) \mapsto (c \otimes d) \otimes (c \otimes d)$ and hence $c \otimes d$ is grouplike if $c, d$ are.

So $C \oplus D$ and $C \otimes D$ are grouplike.
Problem Statement

Let \((C, \Delta, \varepsilon)\) be a coalgebra over the field \(K\). Given grouplike elements \(g, h \in C\), and element \(c \in C\) is called \((g, h)\)-primitive if

\[
\Delta(c) = g \otimes c + c \otimes h.
\]

1. Show that if \(c\) is \((g, h)\)-primitive, then \(\varepsilon(c) = 0\).
2. Let \(V\) be the set of \((g, h)\)-primitive elements of \(C\). Show that:
   - \(V\) is a subspace of \(C\);
   - \(D = V + Kg + Kh\) is a subcoalgebra of \(C\);
   - \(V\) is a coideal of \(D\);
   - and \(D/V\) is a grouplike coalgebra.
Recall that from Problem 3, we have that $\varepsilon(g) = 1$ for any grouplike $g \in C$.
Suppose $c \in C$ is $(g, h)$-primitive. Notice that

$$(\text{id}_C \otimes \varepsilon)(\Delta(c)) = (\text{id}_C \otimes \varepsilon)(g \otimes c + c \otimes h) = g \otimes \varepsilon(c) + c \otimes \varepsilon(h).$$

From the properties of coalgebras, it follows that

$$c = \varepsilon(c)g + \varepsilon(h)c = \varepsilon(c)g + c \iff \varepsilon(c)g = 0.$$ 

Since $g$ is grouplike, $g \neq 0$ so $\varepsilon(c) = 0$. \qed
$V$ is a subspace of $C$

Clearly $0 \in C$ is $(g, h)$-primitive, so we only have to show closure under addition and scalar multiplication. Suppose $c_1, c_2 \in V$ and $k \in K$. Then

$$
\Delta(c_1 + c_2) = \Delta(c_1) + \Delta(c_2) \\
= g \otimes c_1 + c_1 \otimes h + g \otimes c_2 + c_2 \otimes h \\
= g \otimes (c_1 + c_2) + (c_1 + c_2) \otimes h
$$

so $c_1 + c_2 \in V$. We also have that

$$
\Delta(kc_1) = k\Delta(c_1) = k(g \otimes c_1 + c_1 \otimes h) = g \otimes (kc_1) + (kc_1) \otimes h
$$

so $kc_1 \in V$. \qed
\[ D = V + Kg + Kh \text{ is a subcoalgebra of } C \]

Recall that if \( g \in G \) is grouplike, then \( \Delta(g) = g \otimes g \).
Since \( D \) is a subspace of \( C \), we only have to check that \( \Delta(D) \subset D \otimes D \).
Suppose \( d \in D \). Then we can write \( d = c + k_1g + k_2h \) for \( c \in V \) and \( k_1, k_2 \in K \). Notice that

\[
\Delta(d) = g \otimes c + c \otimes h + k_1(g \otimes g) + k_2(h \otimes h).
\]

Since \( g, h, c \in D \), we have that \( \Delta(d) \in D \otimes D \).
Recall that a subspace $l \subset C$ is called a coideal of $\varepsilon(l) = 0$ and \[ \Delta(l) \subset l \otimes C + C \otimes l. \]

From part 1, we have that $\varepsilon(V) = 0$. Suppose $c \in V$. Then \[ \Delta(c) = c \otimes h + g \otimes c. \]

Since $g, h \in D$, we are done.
$D/V$ is grouplike

To show this, we are going to show that $D \cong K$ as $K$-vector spaces. First we show that $V \cap (Kg + Kh) = K(g - h)$. Suppose that $c \in V \cap (Kg + Kh)$. Then we can write $c = k_1 g + k_2 h$. Therefore

$$0 = \varepsilon(c) = k_1 \varepsilon(g) + k_2 \varepsilon(h) = k_1 + k_2$$

so $k_2 = -k_1$ so $c \in K(g - h)$. Now notice that for any $k \in K$,

$$\Delta(kg - kh) = g \otimes (kg) + (-kh) \otimes h$$

$$= g \otimes (kg) - k(g \otimes h) + k(g \otimes h) + (-kh) \otimes h$$

$$= g \otimes (kg - kh) + (kg - kh) \otimes h$$

so $kg - kh \in V \cap (Kg + Kh)$.
**D/V is grouplike**

Consider the map

\[
\phi : D \rightarrow K
\]

\[
c + k_1g + k_2h \mapsto k_1 + k_2.
\]

First we show that \(\phi\) is well-defined. Suppose

\[
d_k := c_k + k_1g + k_2h = c_t + t_1g + t_2h =: d_t.
\]

Then

\[
(k_1 - t_1)g + (k_2 - t_2)h \in V \cap (Kg + Kh) = K(g - h).
\]

Therefore

\[
k_1 - t_1 + k_2 - t_2 = 0 \iff k_1 + k_2 = t_1 + t_2.
\]

Thus

\[
\phi(d_k) = k_1 + k_2 = t_1 + t_2 = \phi(d_t).
\]
Clearly $\phi$ is $K$-linear and surjective. Also since

$$\varepsilon(c + k_1g + k_2g) = k_1 + k_2 = \varepsilon(\phi(c + k_1g + k_2g))$$

and

$$(\phi \otimes \phi)(\Delta(c + k_1g + k_2h)) = k_1 \otimes 1 + k_2 \otimes 1 = \Delta(k_1 + k_2)$$

$\phi$ is a morphism of coalgebras. Thus we only have to show that $\ker \phi = V$. If $c \in V$, then $c = c + 0g + 0h$ so $\phi(c) = 0$.

Now suppose that $d \in \ker \phi$. We can write $d = c + k_1g + k_2h$ for $c \in V$ and $k_1, k_2 \in K$. Then $k_1 + k_2 = 0$ so $k_2 = -k_1$. Therefore $d \in V + K(g - h)$. Since $K(g - h) \subset V$, we have that $d \in V$ and so $\ker \phi = V$.

By the first isomorphism theorem, it follows that $D/V \cong K$. Thus any non-zero element forms a basis for $D/V$ so $\{g + V\}$ is a $K$-basis of grouplike elements.
Let $C = k / N$ be the monoid coalgebra of $N = \{ x^1, x^2, \ldots \}$.

(a) Show that $C^* \cong k[\text{Id}]$.

(b) Find all subalgebras of $C$.

Proof: Let $f : C^* \to k[\text{Id}]$. WTS that $C$ is an algebra map.

\[ f \mapsto \sum_{i=0}^{n} f(x^i)y^i \]

\[ \Delta^*(f \circ g)(x^m) = \sum_{i=0}^{\min(m,n)} f(x^i) \cdot g(x^{m-i}) \] is the coefficient of $x^i$ of $(g)_{\text{Id}}(g)$ on the degree $n$ term.

\[ \Rightarrow C \text{ is an algebra homomorphism and is a bijection.} \]

(b) Define $k[I]$ as the subspace of $k / N$ with basis $\{ x^1, x^2, \ldots \}$.

Claim: $k[I]$ are all the subcoalgebras of $k / N$.

Proof: $V$ a vector space $w \in V$, then $W < (W^1)^1$ by definition.

Let $D \subseteq C$ be a subcoalgebra, then $D \subseteq C^*$ is an ideal.

$k[\text{Id}]$ is PID and $(y^m)$ are the only ideals.

then $D^1 = (y^n)$ for some $n$ and $(y^n)^1 = k[I - 1]$, for any $f \in (y^n)^1$, $f(x^i) = 0$ for \[ i < n \]

and suppose $\sum_{i=0}^{n} f(x^i) \in (y^n)^1$ with $m > n$.

then let $\lim_{m \to n}$, we have $y^n : y^m (x^{m-n}) = 1 \equiv 0$

\[ \Rightarrow (y^n)^1 = k[I - 1] \]

$k[I]$ is a subcoalgebra, $\Delta(x^n) = \sum x^i \otimes x^j \in k[I] \otimes k[I]$

We find all the subcoalgebras of $k[I]$, which is fol.

cisider $(k[I]^*) \cong k[y^1, y^2, \ldots , y^n]$

\( \{ 1, x, \ldots, x^n \} \) is a basis for $k[I]$.

Let $\{ 1', x', \ldots, x'^n \}$ be the dual basis.

\[ \Delta^*(x'^{i} \otimes x'^{j}) = (x'^{i+j}), \text{ then } (k[I]^*)^* \text{ is a n-d k-algebra generated by } x', \text{ and } (x')^{n+1} = 0.\]
all ideals of \( k[y]/(y^n) \) look like \( (y^k) \) for integer \( k \neq 0 \)

\((y^k)^2 \subseteq \kappa^k\)
Problem 7.
Recall that for a coalgebra \((C, \Delta, \varepsilon)\), we have showed that the dual \((C^*, \Delta^*, \varepsilon^*)\) is an algebra.

(a) Suppose that \(\phi: C \to D\) is a homomorphism of colagebras. Show that the dual map \(\phi^*: D^* \to C^*\) is a homomorphism of algebras.

Solution: Note that \(\phi\) being a homomorphism of colagebras implies

\[(\phi \otimes \phi) \circ \Delta_C = \Delta_D \circ \phi \quad \text{and} \quad \varepsilon_C = \varepsilon_D \circ \phi.
\]

Thus

\[
\phi^*(1_{D^*}) = \phi^*(\varepsilon_D) = \varepsilon_D \circ \phi = \varepsilon_C = 1_{C^*}
\]

and

\[
\phi^*(fg) = (\phi \otimes \phi)^*(\Delta_D(f \otimes g)) = ((\phi \otimes \phi) \circ \Delta_C)^*(f \otimes g) = (\phi \otimes \phi) \circ \Delta_C(f \otimes g) = \phi^*(f) \phi^*(g)
\]

for \(f \in C^*, g \in D^*\). Here (*) is true since

\[
[(\phi \otimes \phi) \circ \Delta_C]^*(f \otimes g)](c) = (f \otimes g)\left(\sum (\phi(c(1)) \otimes \phi(c(2)))\right)
\]

\[
= \sum f(\phi(c(1))) \otimes g(\phi(c(2)))
\]

\[
= \sum \phi^*(f(c(1))) \otimes \phi^*(g(c(2)))
\]

\[
= (\phi^* f \otimes \phi^* g) \circ \Delta_C(c)
\]

\[
= [\Delta^*_C \circ (\phi^* f \otimes \phi^* g)](c)
\]

for \(f \in C^*, g \in D^*, c \in C\). Hence \(\phi^*\) is a homomorphism of algebras.

(b) Suppose that \(C\) and \(D\) are colagebras and consider the map \(\psi: C^* \otimes D^* \to (C \otimes D)^*\) given by \(\psi((f \otimes g)(v \otimes w) = f(v)g(w)\) for \(f \in V^*, g \in W^*, v \in V, w \in W\). Show that \(\psi\) is a homomorphism of algebras.

Solution: Note that

\[
\psi(1_{C^* \otimes D^*}) = \psi(\varepsilon_C \otimes \varepsilon_D) = \varepsilon_{C \otimes D} = 1_{(C \otimes D)^*}
\]

since \(\varepsilon_{C \otimes D}(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d)\). Now note that for \(f, g \in C^*, \alpha, \beta \in D^*, \)

\[
m_{C^* \otimes D^*} = (m_{C^*} \otimes m_{D^*}) \circ \tau_{23} = (\Delta_C^* \otimes \Delta_D^*) \circ \tau_{23}
\]

implies that

\[
\psi((f \otimes \alpha)(g \otimes \beta)) = \psi \circ m_{C^* \otimes D^*}((f \otimes \alpha) \otimes (g \otimes \beta))
\]

\[
= \psi \circ (\Delta_C^* \otimes \Delta_D^*) \circ \tau_{23}((f \otimes \alpha) \otimes (g \otimes \beta))
\]

\[
= \psi(\Delta_C^*(f \otimes g) \otimes \Delta_D^*(\alpha \otimes \beta))
\]

\[
= \psi(\Delta_C^*(f \otimes g) \otimes \Delta_D^*(\alpha \otimes \beta)), \quad (1)
\]
and
\[ m_{(C \otimes D)^*} = \Delta^*_{C \otimes D} = (\tau_{23} \circ (\Delta_C \otimes \Delta_D))^* \]
implies that
\[
\psi(f \otimes \alpha)\psi(g \otimes \beta) = m_{(C \otimes D)^*}(\psi(f \otimes \alpha) \otimes \psi(g \otimes \beta)) \\
= (\tau_{23} \circ (\Delta_C \otimes \Delta_D))^* (\psi(f \otimes \alpha) \otimes \psi(g \otimes \beta)) \\
= (\psi(f \otimes \alpha) \otimes \psi(g \otimes \beta)) \circ \tau_{23} \circ (\Delta_C \otimes \Delta_D). \tag{2}
\]
Now, for \(f, g \in C^*, \alpha, \beta \in D^*, c \in C, d \in D\), (1) implies that
\[
[\psi((f \otimes \alpha)(g \otimes \beta))] (c \otimes d) = [\psi(\Delta_C^*(f \otimes g) \otimes \Delta_D^*(\alpha \otimes \beta))] \\
= [\Delta_C^*(f \otimes g)](c)[\Delta_D^*(\alpha \otimes \beta)](d) \\
= \left( \sum_{c} f(c_{(1)}) \otimes g(c_{(2)}) \right) \left( \sum_{d} \alpha(d_{(1)}) \otimes \beta(d_{(2)}) \right) \\
= \left( \sum_{c} f(c(1))g(c(2)) \right) \left( \sum_{d} \alpha(d_{(1)})\beta(d_{(2)}) \right) \\
= \sum_{c,d} f(c(1))g(c(2))\alpha(d_{(1)})\beta(d_{(2)}),
\]
and (2) implies that
\[
[\psi(f \otimes \alpha)\psi(g \otimes \beta)] (c \otimes d) = (\psi(f \otimes \alpha) \otimes \psi(g \otimes \beta)) \circ \tau_{23} \circ (\Delta_C \otimes \Delta_D)(c \otimes d) \\
= (\psi(f \otimes \alpha) \otimes \psi(g \otimes \beta)) \\
\quad \circ \tau_{23} \left( \left( \sum_{c} c(1) \otimes c(2) \right) \left( \sum_{d} d_{(1)} \otimes d_{(2)} \right) \right) \\
= (\psi(f \otimes \alpha) \otimes \psi(g \otimes \beta)) \\
\quad \circ \tau_{23} \left( \sum_{c,d} c(1) \otimes c(2) \otimes d_{(1)} \otimes d_{(2)} \right) \\
= \left[ \psi(f \otimes \alpha) \otimes \psi(g \otimes \beta) \right] \left( \sum_{c,d} c(1) \otimes d_{(1)} \otimes c(2) \otimes d_{(2)} \right) \\
= \sum_{c,d} \left( \psi(f \otimes \alpha)(c(1) \otimes d_{(1)}) \otimes \left( \psi(g \otimes \beta)(c(2) \otimes d_{(2)}) \right) \right) \\
= \sum_{c,d} f(c(1))\alpha(d_{(1)}) \otimes g(c(2))\beta(d_{(2)}) \\
\quad \sum_{\epsilon k \otimes k = k} \epsilon_{k} \otimes k = k \\
= \sum_{c,d} f(c(1))\alpha(d_{(1)})g(c(2))\beta(d_{(2)}).
\]
Noting that the two expressions agree (since \(k\) is commutative), we see that
\[
\psi((f \otimes \alpha)(g \otimes \beta)) = \psi(f \otimes \alpha)\psi(g \otimes \beta).
\]
Hence \(\psi\) is a homomorphism of algebras.