1. **Algebras and coalgebras**

1.1. **Algebra basics.** Let $k$ be a field. Recall the following definition:

**Definition 1.1.** A $k$-algebra is a ring with unit $A$ which is also a $k$-vector space, such that $\lambda \cdot (ab) = (\lambda \cdot a)b = a(\lambda \cdot b)$ for all $\lambda \in k$ and $a, b \in A$, where $\cdot$ indicates the scalar multiplication.

Given a $k$-algebra $A$, there is a map $u : k \to A$ given by $\phi(\lambda) = \lambda \cdot 1$. One may check that $u$ is a ring homomorphism whose image is contained in the center of $A$. Conversely, if $A$ is a ring and $u : k \to A$ is a ring homomorphism whose image lies in the center of the ring $A$, then $A$ becomes a $k$-algebra with scalar multiplication $\lambda \cdot a = u(\lambda)a$. Moreover, since we are only concerned with algebras over a field in these notes, the homomorphism $u$ must be injective and so we can identify $k$ with its image under $u$. Thus we can also think of an algebra $A$ as a ring containing a copy of the field $k$ inside its center.

**Example 1.2.** Let $M$ be a monoid, that is a set with an associative product that has an identity element. We define the *monoid algebra* $kM$ to be the a vector space with basis $M$, where the product is induced by the product of $M$ extended linearly. More formally, we may write an arbitrary element of $kM$ as $\sum_{m \in M} a_m m$, where if $M$ is infinite then all but finitely many $a_m$ are 0. Then

$$\left(\sum_{m \in M} a_m m\right)\left(\sum_{m \in M} b_m m\right) = \sum_{m \in M, n \in M} a_m b_n mn,$$

where $mn$ is the product in the monoid $M$. Note that the identity element of $kM$ is the identity element $1_M$ of the monoid $M$.

A group is just a monoid for which every element has a multiplicative inverse. We will be especially interested in the special case of group algebras below.

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**Example 1.3.** As a special case of the preceding example, let $M$ be the monoid $\mathbb{N}$ of nonnegative natural numbers under addition, which we write multiplicatively as $\{x^0, x^1, x^2, x^3 \ldots \}$, so that $x^i x^j = x^{i+j}$. Then clearly $kM$ is isomorphic to the algebra $k[x]$ of polynomials in one variable. If instead we take the group $\mathbb{Z}$ of all integers under addition, then we get the algebra of Laurent polynomials $k\mathbb{Z} \cong k[x, x^{-1}]$.

You should be familiar with many more examples of algebras from your study of ring and module theory. In this course many of the main results will concern *finite dimensional algebras*, that is $k$-algebras $A$ for which $\dim_k A < \infty$. If $M$ is a finite monoid then $kM$ is such an example. Another simple example is the ring $M_n(k)$ of $n \times n$ matrices with entries in $k$.

We will make heavy use of tensor products in this course, but primarily tensor products over a field, which are especially easy to understand. We won’t review the general definition and theory of tensor products here. Recall, however, that if $V$ and $W$ are $k$-vector spaces, with respective $k$-bases $\{v_i | i \in I\}$ and $\{w_j | j \in J\}$, then the tensor product $V \otimes_k W$ has a $k$-basis of pure tensors $\{v_i \otimes w_j | (i, j) \in I \times J\}$. This gives a very explicit way of thinking of a tensor product over a field, though for some purposes it is better to rely on the universal property of the tensor product rather than thinking in terms of bases.

**Example 1.4.** Suppose that $A$ and $B$ are $k$-algebras. Then the direct sum $A \oplus B$ (as vector spaces) is naturally a $k$-algebra, with product $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$. Note that the scalar multiplication is $\lambda \cdot (a, b) = (\lambda a, \lambda b)$.

The tensor product $A \otimes_k B$ is also a $k$-algebra, with product induced by the extending linearly the product on pure tensors given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1a_2 \otimes b_1b_2)$, and with scalar multiplication $\lambda \cdot (a \otimes b) = (\lambda a \otimes b) = (a \otimes \lambda b)$.

1.2. **Diagrammatic definition of an algebra.** We would like to “dualize” the definition of an algebra. In order to do this we need to first express the definition of algebra in terms of commutative diagrams.

First, note that if $A$ is a $k$-algebra, then the map $A \times_k A \to A$ given by $(a, b) \mapsto ab$ is bilinear and balanced. We refer to bilinear and balanced over $k$ as “$k$-bilinear” from now on. Thus by the universal property of the tensor product, we get a unique $k$-linear map $m : A \otimes_k A \to A$ with $a \otimes b \mapsto ab$ which we refer to as the *multiplication map* of the
algebra. As noted earlier, we can also think of the $k$-vector space structure in terms of a ring homomorphism $u : k \to A$ with image in the center of $A$, which we refer to as the unit map.

In the following result, we use that there are canonical isomorphisms $k \otimes_k V \cong V$ and $V \otimes_k k \cong V$ for any $k$-vector space $V$, and take these as identifications. From now on, because almost all tensor products will be over the field $k$, we write $\otimes_k$ as $\otimes$ when there is no chance of confusion. We write $\text{id}_S$ for the identity map $S \to S$ of any set $S$, or sometimes just $\text{id}$ if the set $S$ is clear.

**Lemma 1.5.** Suppose that $A$ is a $k$-vector space together with $k$-linear maps $m : A \otimes A \to k$ and $u : k \to A$. These maps give $A$ the structure of a $k$-algebra for which $m$ and $u$ are the multiplication and unit maps, if and only if the following two diagrams are commutative:

$$
\begin{array}{ccc}
A \otimes k &= A &= k \otimes A \\
\downarrow \text{id}_A \otimes u & \downarrow \text{id}_A & \downarrow u \otimes \text{id}_A \\
A \otimes A & \overset{m}{\longrightarrow} & A \\
\end{array}
$$

$$
\begin{array}{ccc}
A \otimes A \otimes A & \overset{m \otimes \text{id}_A}{\longrightarrow} & A \otimes A \\
\downarrow \text{id}_A \otimes m & & \downarrow m \\
A \otimes A & \overset{m}{\longrightarrow} & A
\end{array}
$$

**Proof.** The commutativity of the first diagram says that $1_A = u(1_k)$ is a unit element of $A$, and the commutativity of the second diagram says precisely that the product given by $m$ is associative.

The top diagram also says that for $\lambda \in k, a \in A$, $\lambda \cdot a = (\lambda \cdot 1_A) a = a(\lambda \cdot 1_A)$, so that $k1_A = u(k)$ is in the center of $A$. \qed

We sometimes refer to an algebra by the triple $(A, m, u)$ of the $k$-vector space $A$ and the two maps $m$ and $u$ that define the algebra structure.

### 1.3. Coalgebras

The definition of a coalgebra is made by reversing the arrows in the diagrams in Lemma 1.5. This leads to a notion that seems much less intuitive than an algebra at first, but we will see that there are many examples.

**Definition 1.6.** Suppose that $C$ is a $k$-vector space together with $k$-linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$. Then $C$ is called a coalgebra, and the maps $\Delta$ and $\epsilon$ are called the
composite (or coproduct) and counit respectively, if the following two diagrams are commutative:

\[
\begin{align*}
C \otimes C & \xrightarrow{\Delta} C \\
C \otimes k & = C = k \otimes C \\
\end{align*}
\]

\[
\begin{align*}
C & \xrightarrow{\Delta} C \otimes C \\
\Delta \otimes \text{id}_C & = \text{id}_C \otimes \Delta \\
\end{align*}
\]

We sometimes refer to a coalgebra by the triple \((C, \Delta, \epsilon)\) of the \(k\)-vector space \(C\) and the two maps \(\Delta\) and \(\epsilon\) that define the coalgebra structure.

Many common examples of product operations defining algebras involve combining two elements in a natural way such as multiplication of numbers or composition of functions. Conversely, many natural coproducts take an element and pull it apart into two pieces in all possible ways, summing over the possibilities.

**Example 1.7.** Let \(M\) be a monoid with the property that for all \(m \in M\), there are finitely many pairs \((n, p) \in M \times M\) such that \(np = m\). Let \(kM\) be the vector space with basis given by the elements of \(M\), and define a coproduct via \(\Delta(m) = \sum_{(n, p) \in M^2} n \otimes p\), then extending linearly to define a \(k\)-linear map \(\Delta : kM \to kM \otimes kM\). Define \(\epsilon : kM \to k\) by \(\epsilon(1_M) = 1\) and \(\epsilon(m) = 0\) for all \(1_M \neq m \in M\), again extending linearly.

We claim that \((kM, \Delta, \epsilon)\) is a coalgebra, which we refer to as the **monoid coalgebra** of \(M\). Since both \(\Delta\) and \(\epsilon\) are defined on the objects of \(M\) and then extended linearly to \(kM\), it is easy to see that to check the necessary diagrams hold it is enough to check they commute when starting with an element \(m \in M\). This is because all of the maps in the diagrams are \(k\)-linear.

For the top diagram we must check that \(\sum_{np=m} \epsilon(n)p = m\). This is clear since every summand is zero except for the one with \(n = 1_M, p = m\). Similarly \(\sum_{np=m} n \epsilon(p) = m\). Next, we note that

\[
(\Delta \otimes \text{id}_C) \circ (\Delta)(m) = (\Delta \otimes 1_C) \sum_{np=m} n \otimes p = \sum_{qr=n} \sum_{np=m} q \otimes r \otimes p = \sum_{qrp=m} q \otimes r \otimes p.
\]
Similarly,

\[(1_C \otimes \Delta) \circ (\Delta)(m) = (1_C \otimes \Delta) \sum_{np=m} n \otimes p = \sum_{st=p} \sum_{np=m} n \otimes s \otimes t = \sum_{nst=m} n \otimes s \otimes t.\]

So we see that these are the same.

**Example 1.8.** As a special case of the previous example, consider the multiplicative monoid \(\{x^0, x^1, x^2, \ldots\}\) which is isomorphic to the nonnegative integers under addition. The coalgebra defined above has coproduct and counit given by \(\Delta(x^n) = \sum_{i+j=n} x^i \otimes x^j\) and \(\epsilon(x^n) = \delta_{0n}\). Here \(\delta_{ij}\) is the “Kronecker delta”, where \(\delta_{ij} = 1\) if \(i = j\) and \(\delta_{ij} = 0\) if \(i \neq j\).

**Example 1.9.** The field \(k\) is a coalgebra in a canonical way, where \(\Delta : k \rightarrow k \otimes k = k\) and \(\epsilon : k \rightarrow k\) are both the identity maps under the natural identification of \(k \otimes_k k\) with \(k\). This is called the **trivial coalgebra**.

**Example 1.10.** Let \(A = M_n(k)\) be the \(k\)-algebra of \(n \times n\)-matrices with entries in \(k\). For \(1 \leq i, j \leq n\), let \(e_{ij} \in A\) be the matrix with a 1 in row \(i\) and column \(j\) and a zero in all other entries. Then it is easy to check that \(e_{ij}e_{st} = \delta_{js}e_{it}\). The elements \(\{e_{ij} | 1 \leq i, j \leq n\}\) are traditionally called **matrix units** (though they are not units in the ring \(A\)) and they clearly form a basis for \(A\).

Now we can give \(M_n(k)\) a coalgebra structure by defining \(\Delta(e_{ij}) = \sum_{s=1}^n e_{is} \otimes e_{sj}\) and \(\epsilon(e_{ij}) = \delta_{ij}\). We leave it to the reader to show that \((A, \Delta, \epsilon)\) is a coalgebra.

**Example 1.11.** Let \(S\) be any set and let \(kS\) be the vector space with basis \(S\). The **grouplike** coalgebra on this set is \((kS, \Delta, \epsilon)\) where \(\Delta(s) = s \otimes s\) and \(\epsilon(s) = 1\) for all \(s \in S\). It is easily checked to be a coalgebra.

More generally, in any coalgebra \((C, \Delta, \epsilon)\), an element \(c \in C\) with \(\Delta(c) = c \otimes c\) and \(\epsilon(c) = 1\) is called a **grouplike element**. Thus the grouplike coalgebra has a \(k\)-basis of grouplike elements.

**Example 1.12.** Suppose that \((C, \Delta_C, \epsilon_C)\) and \((D, \Delta_D, \epsilon_D)\) are coalgebras over \(k\). The direct sum \(C \oplus D\) is a naturally a coalgebra, as follows. Note that we have a canonical isomorphism

\[(C \oplus D) \otimes (C \oplus D) \cong (C \otimes C) \oplus (C \otimes D) \oplus (D \otimes C) \oplus (D \otimes D).\]
Then we define the coproduct $\Delta$ of $C \oplus D$ as the composition

$$C \oplus D \xrightarrow{\Delta_{C \oplus D}} (C \otimes C) \oplus (D \otimes D) \xrightarrow{\iota} (C \oplus D) \otimes (C \oplus D),$$

where $\iota$ is the natural inclusion. Defining also $\epsilon((c, d)) = \epsilon_C(c) + \epsilon_D(d)$, it is straightforward to check that $((C \oplus D), \Delta, \epsilon)$ is a coalgebra. We can also describe the coproduct of $C \oplus D$ more explicitly, as follows. For $c \in C$ we write $\Delta_C(c) = \sum_{i=1}^m c_{i,1} \otimes c_{i,2}$ and similarly for $d \in D$ we write $\Delta_D(d) = \sum_{j=1}^n d_{j,1} \otimes d_{j,2}$. Then

$$\Delta((c, d)) = \sum_{i=1}^m (c_{i,1}, 0) \otimes (c_{i,2}, 0) + \sum_{j=1}^n (0, d_{j,1}) \otimes (0, d_{j,2}).$$

The tensor product $C \otimes D$ also has a coalgebra structure. The coproduct $\Delta$ is the composition

$$C \otimes D \xrightarrow{\Delta_{C \otimes D}} C \otimes C \otimes D \otimes D \xrightarrow{\tau_{23}} C \otimes D \otimes C \otimes D$$

where $\tau_{23}$ is the function that switches the 2nd and 3rd tensorands. The counit is given by $\epsilon(c \otimes d) = \epsilon(c)\epsilon(d)$. Again, if we wish, we can write the formula for the coproduct $\Delta$ explicitly on elements as

$$\Delta(c \otimes d) = \sum_{i=1}^m \sum_{j=1}^n (c_{i,1} \otimes d_{j,1}) \otimes (c_{i,2} \otimes d_{i,2}).$$

1.4. **Sweedler notation.** If we need to write the action of a coproduct on an element of a coalgebra $C$ in coordinates, formally we get something like the notation $\Delta(c) = \sum_{i=1}^m c_{i,1} \otimes c_{i,2}$ we wrote in the previous example. This is awkward in several ways: the number of summands $m$ depends on the element $c$ in general; the double indexing is ugly; and if we had to apply $(\Delta \otimes 1_C)$ to $\Delta(c)$ even more indices would appear.

Moss Sweedler invented a simplified notation which is in wide use. One just writes $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$. The number of summands is undetermined, and one does not index the elements at all, other than indicating their positions in the tensor product. This may be confusing at first, but once one gets used to this notation it adds great clarity to proofs in which one needs to work with the action of the coproduct on an arbitrary element.

By the axioms of a coalgebra, $(\Delta \otimes 1_C) \circ \Delta = (1_C \otimes \Delta) \circ \Delta$. Sometimes this map is written as $\Delta^{(2)} : C \rightarrow C \otimes C \otimes C$. In Sweedler notation, applying the first operation to $c$ gives

$$(\Delta \otimes 1_C) \circ \Delta(c) = (\Delta \otimes 1_C)(\sum c_{(1)} \otimes c_{(2)}) = \sum_{i=1}^m c_{i(1)}(1) \otimes c_{i(2)}(1) \otimes c_{(2)}.$$
On the other hand,

\[(1_C \otimes \Delta) \circ \Delta(c) = (1_C \otimes \Delta)(\sum c_{(1)} \otimes c_{(2)}) = \sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}.\]

Now the expressions \(\sum c_{(1)}(1) \otimes c_{(1)}(2) \otimes c_{(2)}\) and \(\sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}\) must represent the same element of \(C \otimes C \otimes C\). Sweedler’s notation simplifies this further, and represents this element as \(\Delta^{(2)}(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}\). In this way, double Sweedler indices can be avoided. Notice that the indices just refer to the positions of the tensorands and not specific elements.

In Sweedler notation, the axiom of the counit \(\epsilon\) takes the following form: for all \(c \in C\), one has \(c = \sum \epsilon(c_{(1)})c_{(2)} = \sum c_{(1)}\epsilon(c_{(2)})\).

There is an alternative notation which simplifies even further by avoiding the parentheses and just writes \(\Delta(c) = \sum c_1 \otimes c_2\). If one is feeling especially lazy, one even omits the sum and writes \(\Delta(c) = c_1 \otimes c_2\). This is a bit dangerous since one must constantly remember that there is an implicit sum and \(c_1 \otimes c_2\) does not stand for a pure tensor. The author is a fan of the sumless notation in contexts when there are many complicated formulas, but will try to avoid it in these notes.

We will begin to use the Sweedler notation and the reader will soon see how it works in practice.

1.5. **Basic properties of vector space duals.** In this section we give reminders about the basic properties of vector space duals. Let \(V\) be a vector space over the field \(k\). The dual space is \(V^* = \text{Hom}_k(V, k)\), that is, the collection of all linear transformations \(f : V \to k\). Such an \(f\) is sometimes called a linear functional. \(V^*\) is naturally itself a vector space with pointwise operations: if \(f, g \in V^*\) and \(\lambda \in k\), then \(f + g \in V^*\) where \([f + g](v) = f(v) + g(v)\) for \(v \in V\), and \(\lambda f \in V^*\) where \([\lambda f](v) = \lambda f(v)\).

Suppose that \(V\) is finite dimensional, with \(k\)-basis \(v_1, \ldots, v_n\). Then we define the dual basis of \(V^*\) to be \(\{v_1^*, \ldots, v_n^*\}\), where \(v_i^*(v_j) = \delta_{ij}\) (here \(\delta_{ij}\) is the Kronecker delta). It is easy to check that this is a basis of \(V^*\); in particular, \(\dim_k V^* = n = \dim_k V\). Since the vector spaces \(V\) and \(V^*\) have the same dimension, there is a vector space isomorphism between them, but there is no canonical isomorphism. In particular, the isomorphism \(V \to V^*\) with \(v_i \mapsto v_i^*\) is highly dependent on the original choice of basis.
If $V$ is infinite dimensional over $k$, then $V$ and $V$ cannot be isomorphic as vector spaces, as the dimension of the vector space $V^*$ has a larger cardinality than the dimension of $V$, though we don’t prove that here.

We write $V^{**}$ for the double dual $(V^*)^* = \text{Hom}_k(\text{Hom}_k(V, k), k)$. There is a canonical vector space map $i : V \to (V^*)^*$ given by $v \mapsto e_v$, where $e_v \in V^{**}$ is “evaluation at $v$”, that is $e_v(f) = f(v)$ for $f \in V^*$. The map $i$ is always injective. If $V$ is finite dimensional over $k$, then we have $\dim_k V = \dim_k V^* = \dim_k V^{**}$ as observed above. Thus in this case $i$ is an isomorphism since it is an injective linear transformation between vector spaces of the same finite dimension. We see that $V$ and $V^{**}$ are canonically isomorphic in this case. If $V$ is infinite dimensional over $k$, then $i$ cannot be an isomorphism, since again the dimension of $V^{**}$ is of a cardinality larger than the dimension of $V$.

Suppose that $\phi : V \to W$ is a linear transformation of $k$-vector spaces. There is an induced “dual” linear transformation of the dual spaces, $\phi^* : W^* \to V^*$, where $[\phi^*(f)](v) = f(\phi(v))$ for $f \in W^*$, $v \in V$. If $i_V : V \to V^{**}$ is the canonical map given above, then there is a commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\phi} & W \\
\downarrow{i_V} & & \downarrow{i_W} \\
V^{**} & \xrightarrow{\phi^{**}} & W^{**}
\end{array}
\]

If $V$ and $W$ are finite dimensional, then $i_V$ and $i_W$ are isomorphisms, so the diagram says that dualizing twice, the map $\phi^{**}$ is essentially the same as $\phi$.

It is useful also to notice how duals interact with the tensor product. For vector spaces $V$ and $W$ we always have a canonical linear transformation $\iota : V^* \otimes W^* \to (V \otimes W)^*$, where $[\iota(f \otimes g)](v \otimes w) = f(v)g(w)$. One may check that $\iota$ is always an injective linear transformation, and that when either $V$ or $W$ is finite dimensional, then $\iota$ is an isomorphism, but not when both $V$ and $W$ are infinite dimensional. This can be proved by choosing bases.

1.6. **Duals of algebras and coalgebras.** Now suppose that $C$ is any coalgebra over $k$. We claim that the dual space $C^*$ has a natural $k$-algebra structure. For $f, g \in C^* = \text{Hom}_k(C, k)$ we define $fg \in C^*$ by $[fg](c) = \sum f(c_{(1)})g(c_{(2)})$, where $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ in Sweedler notation. Note that the counit of $C$ is a linear map $\epsilon : C \to k$, so $\epsilon \in C^*$. We claim that the product above is an associative product on $C^*$ and that $\epsilon$ is the unit element.
We give a direct proof that $C^*$ is an algebra. By its definition, it is clear that the relation sending $(f, g) \mapsto [fg]$ is bilinear, and so induces a linear map $m : C^* \otimes C^* \to C^*$. To check that $m$ defines an associative product, we calculate for $f, g, h \in C^*$ that

$$[(fg)h](c) = \sum f(g(c_1))h(c_2) = \sum f(c_1)g(c_2)h(c_3) = \sum f(c_1)gh(c_2) = [f(gh)](c),$$

where we have used the Sweedler notation for $\Delta^{(2)}$. To check that $\epsilon$ is then a unit for $C^*$ we note that for all $c \in C$ and $f \in C^*$,

$$[\epsilon f](c) = \sum \epsilon(c_1)f(c_2) = f(\sum \epsilon(c_1)c_2)) = f(c),$$

where we have used that $f$ is linear. Thus $\epsilon f = f$. A similar argument shows that $f \epsilon = f$. This shows that $\epsilon$ is the unit element, and so more formally the map $u : k \to C^*$ given by $u(\lambda) = \lambda \epsilon$ is the unit map.