1. Give examples of bialgebras $A$ which show that, unlike the case of Hopf algebras, the dimensions of the spaces of left and right integrals are not tightly controlled. In particular:
   (a). There are bialgebras $A$ where $\int_A^\ell$ has any desired vector space dimension, including $0$ and $\infty$. In particular, an infinite dimensional bialgebra can have a nonzero left integrals.
   (c). The dimensions of $\int_A^\ell$ and $\int_A^r$ can be different, and can differ by an arbitrary number. (Hint: consider monoid algebras).

2. Let $H$ and $H'$ be finite dimensional Hopf algebras.
   (a). Find the spaces of left and right integrals for the Hopf algebra $H \otimes H'$, in terms of the integrals of $H$ and $H'$.
   (b). Show that $H \otimes H'$ is semisimple if and only if $H$ and $H'$ are.
   (c). Show that for general finite dimensional semisimple algebras $A$ and $A'$ over an arbitrary field $k$, $A \otimes_k A'$ might fail to be semisimple.

3. Show that a finite dimensional semisimple Hopf algebra $H$ is unimodular (that is, $\int_H^\ell = \int_H^r$).

4. Let $H$ be a Hopf algebra. Recall that $H^{\text{op}}$ is $H$ with opposite multiplication but the same comultiplication, and $H^{\text{cop}}$ is $H$ with the opposite comultiplication but the same multiplication. $H^{\text{op,cop}}$ has both multiplication and comultiplication reversed.
   (a). Sketch the proof that all of these variations are Hopf algebras, where $S^{-1}$ is the antipode of $H^{\text{op}}$ and $H^{\text{cop}}$ and $S$ is the antipode of $H^{\text{op,cop}}$.
   (b). If $H$ is the 4-dimensional Taft algebra, show that all of these variations are isomorphic to $H$ itself as Hopf algebras, even though $H$ is neither commutative nor cocommutative.

5. Let $H = k\langle g, h, y \rangle/(gh - 1, hg - 1, yg - \zeta gy)$ where $\zeta \in k$. Note that $h$ is the inverse of $g$ in $H$. We can also write this algebra informally as $k\langle g, g^{-1}, y \rangle/(yg - \zeta gy)$.
   (a). Prove that $H$ is a Hopf algebra, where $g$ is grouplike and $y$ is $(1, g)$-primitive.
   (b). Find a formula for $S$ and show that if $\zeta$ is not a root of unity, then $S$ has infinite order.

6. Let $A$ be a Frobenius algebra.
   (a). For a given nondegenerate associative form $(-, -)$ on $A$, show there is a uniquely determined algebra automorphism $\mu : A \to A$, the Nakayama automorphism, such that $(a, b) = (b, \mu(a))$ for all $a, b \in A$. 

(b). For automorphisms $\tau, \sigma$ of $A$, let $^\tau A^\sigma$ be the $(A, A)$-bimodule which is $A$ as a vector space with left and right actions $x * a * y = \tau(x)a\sigma(y)$. Give $A^*$ its standard $(A, A)$-bimodule structure. Show that as $(A, A)$-bimodules, $A^* \cong \mu^* A^1 \cong 1^* A^\mu$.

(c). If $\{-, -\}$ is a different nondegenerate associative form on $A$, leading to a different Nakayama automorphism $\mu'$, then $\mu' = i_x \circ \mu$ for some inner automorphism $i_x : a \mapsto xax^{-1}$ of $A$.

7. Let $H$ be a Hopf algebra over a field $k$ which is algebraically closed.

(a). Suppose that $H$ is cocommutative and cosemisimple. Prove that $H$ is isomorphic to a group algebra. (Hint: every simple cocommutative coalgebra is 1-dimensional).

(b). Let $H$ be finite dimensional and suppose that $H$ is commutative and semisimple. Then $H$ is isomorphic to the dual of the group algebra of a finite group.

(c). Suppose that $\dim_k H = 3$. Show that if $H$ is either semisimple or cosemisimple, then $H \cong k\mathbb{Z}_3$ as Hopf algebras. (In class we will eventually see how to use this to prove problem 3 on the previous homework, by showing that if $\text{char } k = 0$ then $H$ has to be semisimple and cosemisimple.)