Math 207a Winter 2012 Homework exercises 1

Due Friday January 27

You are not expected to complete all exercises. Read the statements of them, and write up and hand in some selection of them which you find interesting. Many of these exercises are taken directly from the Goodearl-Warfield text, but we repeat them here since I do not require you to own the text, and besides I have an older edition of the text than the one currently available. Also, some exercises are more straightforward than others, though I generally won't assign extremely routine ones.

1 Exercises on skew polynomial rings and derivations

1. Let R be a ring with ring-endomorphism $\alpha : R \to R$ and α-derivation $\delta : R \to R$. (a). Prove the following universal property of the skew-polynomial ring: If S is a ring together with a ring homomorphism $\phi : R \to S$ which has an element $\theta \in S$ such that $\theta\phi(r) = \phi(\alpha(r))\theta + \phi(\delta(r))$ in S for all $r \in R$, then there is a unique homomorphism of rings $\widetilde{\phi} : R[x; \alpha, \delta] \to S$ s.t. $\widetilde{\phi}$ restricted to R is ϕ and $\widetilde{\phi}(x) = \theta$.

(b). Suppose that S and T are rings each containing R as a subring. Suppose that S has an element x such that S is a free left R-module on the basis $1, x, x^2, \dots$ such that the formula $xr = \alpha(r)x + \delta(r)$ holds in S for all $r \in R$. Suppose similarly that T has an element y such that T is a free left R-module on the basis $1, y, y^2, \dots$ such that the formula $yr = \alpha(r)y + \delta(r)$ holds in T for all $r \in R$.

Show that there is an isomorphism of rings $\phi : S \to T$ which sends x to y and which restricts to the identity on R. This shows that the ring $R[x; \alpha, \delta]$ which we constructed is uniquely determined up to isomorphism.

2. Let $R[x;\alpha,\delta]$ be a skew-polynomial ring, where α is an *automorphism* of R (i.e. it is a bijective ring homomorphism) and δ is an α-derivation. Show that α^{-1} is an automorphism of the ring R^{op}, that $-\delta\alpha^{-1}$ (this means composition of functions) is an α^{-1}-derivation of R^{op}, and that there is a ring isomorphism $\phi : R[x;\alpha,\delta]^{op} \cong R^{op}[x;\alpha^{-1},-\delta\alpha^{-1}]$ such that $\phi(r) = r$ for all $r \in R$ and $\phi(x) = x$.

3. In class we proved that if D is a division ring and α is an isomorphism of D, then $D[y;\alpha]$ is a left and right principal ideal domain. This exercise shows that this result may fail on the right if α is an endomorphism which is not an isomorphism.

Let k be a field, and let $R = k(x)$ be a rational function field in one variable x over k (i.e. R is the field of fractions of the polynomial ring $k[x]$). Define an endomorphism $\alpha : R \to R$ by the formula $f(x) \mapsto f(x^2)$, i.e. substitute x^2 for x everywhere it appears. This is not an isomorphism since it is not surjective.

Prove that $S = R[y;\alpha]$ is not a right principal ideal domain by showing that the right ideal generated by y and xy is not principal.

4. In class we stated the Hilbert basis theorem for skew-polynomial rings, which says that if R is left (or right) noetherian, α is an automorphism of R and δ is an α-derivation, then $R[x;\alpha,\delta]$ is also left (respectively right) noetherian. This exercise shows that this result may fail, even on both sides, if α is an endomorphism which is not an isomorphism.

Let k be a field, let $R = k[x]$ be the polynomial ring and define an endomorphism $\alpha : R \to R$ by the formula $f(x) \mapsto f(x^2)$. Prove that $S = R[y;\alpha]$ is neither left nor right noetherian.

(On the right, look at the right ideal generated by $xy, yxy, y^2xy, \dots$, and on the left look at the left ideal generated by $x, xy, xy^2, \dots$)

5. Prove the following converse to a theorem proved in class: Let R be a $\mathbb{Q}$-algebra and let $\delta : R \to R$ be a derivation. If $R[x;\delta]$ is a simple ring, then R is δ-simple and δ is an outer derivation.

6. Let $A_1(K)$ be the first Weyl algebra over the field K. Recall that this is the ring $K[x][y;\delta]$ where $\delta : K[x] \to K[x]$ is differentiation with respect to x, i.e.

$$\delta(\sum_{i\geq 0} a_i x_i) = \sum_{i\geq 0} i a_i x^{i-1}.$$

If K has characteristic 0, we proved in class that $A_1(K)$ is a simple ring. Now let K have characteristic $p > 0$. Let Z be the center of A_1, i.e. $Z = \{z \in A_1 | zw = wz \text{ for all } w \in A_1\}$. Show that Z is equal to the K-span of $\{x^{ip}y^{jp} | i, j \geq 0\}$.

Note that given any element $z \in Z$, $A_1 z$ is a 2-sided ideal of A_1, since z is central. Show that $A_1 z \subsetneq A_1$ as long as $z \notin K$. Thus $A_1(K)$ has many proper 2-sided ideals in this characteristic p setting, in stark contrast to the case where K has characteristic 0 and $A_1(K)$ is simple.

7. Let R be a commutative domain, and let K be its field of fractions. Suppose that $\delta : R \to R$ is a derivation. Show that there is a unique extension of δ to a derivation $\widetilde{\delta} : K \to K$. (Hint: use the quotient rule for derivatives).

8. If $F \subseteq K$ is an extension of fields of characteristic 0, show that any derivation δ on F extends (not necessarily uniquely) to some derivation on K. Hint: using Zorn's lemma and problem 7, reduce to the case of a finite algebraic extension $F \subseteq K$, and analyze that case directly (there is a unique extension of the derivation in that case).

2 Exercises on formal triangular matrix rings and bimodules

9. Recall that a module M is *artinian* if it has DCC (descending chain condition) on submodules. A ring R is *left artinian* if it is a artinian as a left R-module by left multiplication. Right artinian is defined analogously.

Let $R = \begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ be a formal triangular matrix ring. Thus, S and T are rings and B is an (S, T)-bimodule. Show that R is left artinian if and only if S and T are left artinian and B is a left artinian S-module. Show that R is right artinian if and only if S and T are right artinian and B is right artinian as a right T-module.

Using this, construct a ring which is left artinian but not right artinian.

10. Let S and T be rings, and let R be the ring $S \otimes_{\mathbb{Z}} T^{op}$ (the tensor product of two rings over $\mathbb{Z}$, say $A \otimes_{\mathbb{Z}} B$, is itself a ring with product defined on pure tensors by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ and extended linearly. Here, A and B are $\mathbb{Z}$-modules in the canonical way, by thinking of their underlying abelian groups.)

Show that there is a one-to-one correspondence between (S, T)-bimodules and left modules over the ring R. (If one wanted to be formal, one could prove that the *category* of (S, T)-bimodules, with the obvious notion of bimodule homomorphisms, is equivalent to the category of left R-modules.) This result is occasionally useful in dealing with bimodules since it allows us to think of a bimodule just as a module in the more usual sense over another ring.