Math 207c Spring 2011 Homework exercises 2

Though you are not required to hand in these exercises, working through some of them (in addition to going over your lecture notes which is also very important) will help you get much more out of the course.

Let K be algebraically closed of characteristic 0. For a representation V on a quiver Q, we are now using the notation V_i for the vector space at the vertex i, and $V_a : V_i \to V_j$ for the linear transformation associated to an arrow $a : i \to j$.

Recall that a quiver Q has finite representation type if there are finitely many distinct indecomposable representations up to isomorphism; otherwise Q has infinite representation type. In class, we proved the half of Gabriel's theorem which states that if a connected quiver has finite representation type, its underlying graph is Dynkin of type A, D, E, using some algebraic geometry. This is the slickest proof, and the geometry of the representation spaces for quivers is an important subject; much recent work on quiver representations is about this. However, it is also interesting to see that one can prove this direction of Gabriel's theorem more directly, without algebraic geometry. In the first three exercises below we indicate a rough outline for how this is done.

1. Suppose that Q is a subquiver of a quiver Q'. Show that there is a functor $F : \operatorname{rep} Q \to \operatorname{rep} Q'$ defined as follows. For a represention V of Q, we define the representation $W = F(V)$ where $W_i = V_i$ if $i \in Q_0$, $W_i = 0$ if $i \in Q'_0 \setminus Q_0$, and on arrows, $W_a = V_a$ if $a \in Q_1$, $W_a = 0$ if $a \in Q'_1 \setminus Q_1$. In other words, we "extend V by 0". I leave it to you to define the action on F on morphisms.

Now using F, prove that if Q has infinite representation type, then so does Q'.

We showed in class that every graph which is not Dynkin or Euclidean contains a Euclidean graph as a subgraph. Thus this exercise implies that to show that a non-Dynkin quiver has a infinite representation type, it suffices to show that quivers with underlying graph Euclidean have infinite representation type.

2. Suppose that a connected quiver Q has underlying graph $\overline{Q}$ which has a cycle; we include the case of a loop or a multiple edge. In other words, we assume that $\overline{Q}$ contains a copy of one of the Euclidean graphs $\widetilde{A}_m$, for any $m \geq 0$. To prove that Q has infinite representation type, it suffices by exercise 1 to prove that a quiver whose underlying graph is $\widetilde{A}_m$ has infinite representation type.

So assume now that $Q_0 = \{1, \ldots, m\}$ and there are m arrows in Q with a_i an arrow (of some orientation) between i and $i+1$ for $1 \leq i \leq m-1$, and a_m an arrow (of some orientation) between m and 1. In this case it is not hard to prove directly that Q has infinite representation type, as follows. For any $\lambda \in K$, define a representation $V = V(\lambda)$ with $V_i = K$ for all i; V_{a_i} the identity map for all $1 \leq i \leq m-1$; and $V_{a_m} : K \to K$ multiplication by λ.

Prove that the representations $V(\lambda)$ are all indecomposable, and that $V(\lambda) \cong V(\mu)$ if and only if $\lambda = \mu$. Since K is infinite, (we assume K algebraically closed), this shows that Q has infinite representation type.

By this exercise, a quiver of finite representation type cannot contain an unoriented cycle (where the notion of cycle includes loops and multiple edges). A connected graph with no cycles is also called a *tree*.

3. Suppose that Q is any connected quiver whose underlying graph $\overline{Q}$ is a tree. Let Q' be a quiver with the same underlying graph as Q. Show that there is a sequence of vertices $i_1, i_2, \ldots, i_m$ such that for each $j \geq 1$, i_j is a sink in $s_{i_{j-1}} \ldots s_{i_1}(Q)$, and where $s_{i_m} s_{i_{m-1}} \ldots s_{i_1}(Q) = Q'$.

This shows that the composition of reflection functors $C^+_{i_m} \ldots C^+_{i_1}$ is a functor from $\operatorname{rep} Q$ to $\operatorname{rep} Q'$. Using this and the properties of reflection functors we proved in class, show that Q has infinite representation type if and only if Q' does.

Given this and the preceding exercises, it now suffices to pick a single convenient orientation for each Euclidean graph of type D and E, and show that that quiver has infinite representation type. This is a more tedious case-by-case analysis, which is too detailed for an exercise, so we will not indicate the proof here. If you are interested, I refer you to Corollary 2.7 on page 259 of the book by Assem, Simson, and Skowronski and the preceding pages.

4. Let Q be a connected quiver whose underlying graph is a tree. Recall that there is a numbering of the vertex set Q_0 of Q with $\{1, 2, \ldots, n\}$ such that for every arrow $a \in Q_1$, $h(a) < t(a)$. (This is just a variation on an exercise from HW 1.) In other words, all arrows point from a larger number to a smaller number. We called this an *admissible* numbering in class. It allows one to define the composition of reflection functors $C^+ = C_n^+ \ldots C_1^+ : \operatorname{rep} Q \to \operatorname{rep} Q$, since i is always a sink of $s_{i-1} \ldots s_1(Q)$. Now let Q be have underlying graph which is Dynkin, and fix an admissible numbering of Q. Recall that for each vertex i, if e_i is the trivial path in the path algebra then KQe_i is an indecomposable projective module. The corresponding representation $P = P(i)$ has $P_j = e_j KQ e_i$ for all j, with P_a being left multiplication by a for any arrow $a : j \to k$.

Show that if $S(i)$ is the simple representation of the quiver $s_i s_{i+1} \ldots s_n(Q)$ supported at vertex i, then $P(i) \cong C_1^- \ldots C_{i-1}^-(S(i))$ and the dimension vector of $P(i)$ is $s_1 \ldots s_{i-1}(\epsilon_i)$ where ϵ_i is the dimension vector of $S(i)$. Also, formulate and prove a similar result for indecomposable injective representations. (Hint: we proved as part of Gabriel's theorem that indecomposable representations are uniquely determined up to isomorphism by their dimension vectors.)

5. As practice in understanding the definition of reflection functors, pick some orientation for the Dynkin graph E_6, number the vertices admissibly, consider the simple representation $S(i)$ supported at a sink i, and calculate what happens when you apply the composition of reflection functors $C = C_6^+ \ldots C_1^+$ repeatedly.

6. Consider the quiver Q with $Q_0 = \{1, 2, 3, 4, 5\}$ and with edges $a : 2 \to 1$, $b : 3 \to 1$, $c : 4 \to 1$, $d : 5 \to 1$. The underlying graph of Q is the Euclidean graph $\widetilde{D}_4$. By Gabriel's theorem, Q has infinite representation type.

Consider the dimension vector $\beta = (2, 1, 1, 1, 1)$ (so the 2 is in the central vertex.) Show by playing around that Q has infinitely many non-isomorphic representations with this dimension vector. In fact β is the vector that we showed spans the radical of the symmetrized Ringel form $(\ ,\ )$. Show that $s_i(\beta) = \beta$ for any i, so applying a reflection functor C_i^+ to any representation with this dimension vector (of any quiver with this underlying graph) just gives another representation of the same dimension vector. If we perform $C_5^+ C_4^+ \ldots C_1^+$ to a representation V with dimension vector β, is the representation we get isomorphic to the original one we started with?