MATH 31AH FALL 2017 MIDTERM 2

Instructions: Show all of your work. You may use the result of one part of a problem in the proof of a later part, even if you do not complete the proof of the earlier part. You may quote basic theorems proved in the textbook or in class, unless the problem tells you not to, or unless the whole point of the problem is to reproduce the proof of such a basic theorem. Do not quote the results of homework exercises. There are 60 points total.

1 (10 pts).

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

(a). Find the reduced echelon form of A. What is the rank of A? (No justification necessary, but show your calculation).

(b). Find explicit bases for the row space $R(A)$, the column space $C(A)$, and the null space $N(A)$. Briefly justify your answers.

2 (10 pts).

Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

Use Gaussian elimination to show that A is invertible and find A^{-1} explicitly. (No justification necessary, but show your calculation).

3 (10 pts). In each case, give positive integers m and n and an example of an $m \times n$ matrix A with the stated properties, or explain why none can exist.

(a). There are vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3 \in \mathbb{R}^m$ such that $A\vec{x} = \vec{b}_1$ has no solution, $A\vec{x} = \vec{b}_2$ has exactly one solution, and $A\vec{x} = \vec{b}_3$ has infinitely many solutions.

(b). $m > n$ and the only solution to the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

(c). $m < n$ and the only solution to the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

4 (10 pts). Let A be an $m \times n$ matrix. Show that there exist $n \times m$ matrices B and C such that $AB = I_m$ and $CA = I_n$ if and only if $n = m$ and A is invertible.

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5 (10 pts). Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be mutually orthogonal vectors in $\mathbb{R}^n$; in other words, $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$.

(a). Show that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$.

(b). Let $T : \mathbb{R}^n \to \mathbb{R}$ be the linear transformation given by $T(\vec{x}) = \vec{x} \cdot \vec{v}_1$. Recall that the kernel of T is $\ker T = \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{0}\}$. What is $\dim \ker T$? Justify your answer.

6 (10 pts).

Let $\mathcal{P}_k$ be the set of all polynomial functions of degree at most k,

$$\mathcal{P}_k = \{f(x) = c_0 + c_1 x + \cdots + c_k x^k | c_0, c_1, \ldots, c_k \in \mathbb{R}\}.$$

We make $\mathcal{P}_k$ into a real vector space with the usual addition and scalar multiplication of functions, namely $[f+g](x) = f(x) + g(x)$ and $[cf](x) = cf(x)$, where $f, g \in \mathcal{P}_k$ and $c \in \mathbb{R}$. *In the following problem you may assume that $\{1, x, x^2, \ldots, x^k\}$ is a basis for $\mathcal{P}_k$, as you proved in homework problem 4.3 #26.*

Let $\mathcal{O}_k$ be the subset of $\mathcal{P}_k$ consisting of odd functions; that is,

$$\mathcal{O}_k = \{f(x) | f \in \mathcal{P}_k \text{ and } f(x) = -f(-x) \text{ for all } x \in \mathbb{R}\}.$$

(a). Show that $\mathcal{O}_k$ is a subspace of $\mathcal{P}_k$.

(b). Find $\dim \mathcal{O}_k$ (the answer depends on k).