More on determinants.

What are the determinants of the elementary matrices?

If \( E = \begin{bmatrix} 1 & \cdots & 0 \\\n & \iddots & \vdots \\
 & 0 & \cdots \end{bmatrix} \) then \( \det(E) = \det(InE) = -1 \) since this is \( In \) with 2 columns switched.

If \( E = \begin{bmatrix} \cdots & c \\\n & \iddots & \vdots \\
 & 0 & \cdots \end{bmatrix} \) then \( \det(E) = c \) since this is \( In \) with one column multiplied by \( c \).

If \( E = \begin{bmatrix} \cdots & j \\\n & \iddots & \vdots \\
 & c \cdots \end{bmatrix} \) then \( \det(E) = 1 \) since this is \( In \) with \( c \) times \( j \)th column added to \( i \)th column.

Now notice: if \( E \) is an elementary matrix, then \( E^T \) is also an elementary matrix of the same type and so \( \det(E) = \det(E^T) \). As a corollary we get
Theorem: If \( A \) is an \( n \times n \) square matrix, then \( \det(A) = \det(A^T) \).

Proof: If \( A \) is singular, i.e., \( \text{rank}(A) < n \), then \( \det(A) = 0 \). But then \( \text{rank}(A^T) = \text{rank}(A) < n \), so \( \det(A^T) = 0 \) as well.

Now assume \( A \) is nonsingular. Then \( A \) has row echelon form \( \begin{pmatrix} I_n \end{pmatrix} \), so \( EA = I_n \) where \( E = E_1 E_2 \cdots E_k \) is a product of elementary matrices \( E_i \). Then \( \det(EA) = \det(I_n) = 1 = \det(E) \det(A) \)

while if we take the transpose, we get

\[
A^T E^T = (I_n)^T = I_n \quad \text{so} \quad \det(A^T) \det(E^T) = 1.
\]

Since \( E^T = (E_k)^T (E_{k-1})^T \cdots (E_1)^T \), we have

\[
\det(E^T) = \det((E_k)^T) \cdots \det((E_1)^T) = \det(E_k) \cdots \det(E_1) = \det(E).
\]
(Since $\det(E_i) = \det(E_i^T)$ for an elementary matrix). So $\det(A) = \det(A^T)$. D.

We still haven't shown that there really is a determinant function that is alternating, multilinear, and normalized, and unique with these properties.

Def. Let $A$ be an $n \times n$ matrix with $n \geq 2$. Given $i$ and $j$ with $1 \leq i < n$ and $1 \leq j \leq n$, we define the $(ij)$th cofactor of $A$ to be $c_{ij} = (-1)^{i+j}$, the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing row $i$ and column $j$ of $A$.

(such an $(n-1) \times (n-1)$ matrix is called a minor of $A$).

Example. $A = \begin{bmatrix} 1 & 4 & -1 \\ 5 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix}$
$c_{2,2} = (-1)^5 \det\begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix} = -7$. 

Thus (expansion by minors along a row)

Let \( n \geq 2 \) and fix some \( i \) with \( 1 \leq i \leq n \).

The function \( D : \text{Mat}_n(\mathbb{R}) \longrightarrow \mathbb{R} \)

\[
A \mapsto \sum_{j=1}^{n} a_{ij} c_{ij}
\]

is alternating, multilinear, and normalized and so is the determinant function.

**Ex.** \( A = \begin{bmatrix} 1 & 2 & 5 & 4 \\ 0 & 1 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix} \).

We calculate \( \det(A) \) by expansion by minors along row 2 (rows with 0's make things easier).

\[
\det(A) = \sum_{j=1}^{4} a_{2j} c_{2j}
\]

\[
= 0 (-1)^{2+1} \begin{vmatrix} 2 & 5 & 4 \\ 1 & 0 & -1 \end{vmatrix} + 1 (-1)^{2+2} \begin{vmatrix} 1 & 5 & 4 \\ 3 & 1 & 0 \end{vmatrix} + 0 (-1)^{2+4} \begin{vmatrix} 1 & 2 & 4 \\ 3 & 4 & 0 \end{vmatrix} + 0 \begin{vmatrix} 3 & 4 & 0 \\ 1 & 1 & -1 \end{vmatrix}
\]

\( \) doesn't matter
\[
\begin{vmatrix}
  1 & 4 \\
  3 & 10 \\
\end{vmatrix}
+ \begin{vmatrix}
  1 & 24 \\
  3 & 40 \\
\end{vmatrix}
\]
\[
= 11 - (4 - 15) + ((-412) - (16 - 6))
\]
\[
= 11 - (-11) + (4) - (10)
\]
\[
= 20.
\]

We can expand by column just as well.

Theorem (expansion by minors along a column)

Let \( n \geq 2 \) and \( j \) be fixed.

The function \( D : M_{n \times n}(\mathbb{R}) \to \mathbb{R} \)
\[
A \mapsto \sum_{i=1}^{n} a_{ij} c_{ij}
\]

is alternating, multilinear, and normalized.

and so is the determinant.

These proofs are a little technical so I won't do the proofs in class. See the text for a proof of expansion by minors along a row.

The column version follows by taking transpose and using \( \det(A) = \det(A^T) \).
Ex. \[
A = \begin{bmatrix}
1 & 2 & 5 & 4 \\
0 & 1 & -1 & 0 \\
3 & 4 & 1 & 0 \\
1 & 1 & 1 & -1
\end{bmatrix}
\]

find \( \det(A) \) by expanding along 4th column.

\[
-4 \left| \begin{array}{ccc}
0 & 1 & -1 \\
3 & 4 & 1 \\
1 & 1 & 1
\end{array} \right| + (-1) \left| \begin{array}{ccc}
0 & 1 & -1 \\
3 & 4 & 1 \\
1 & 1 & 1
\end{array} \right|
\]

\[
= -4(-1) + (-1)(-16) = 20.
\]

Either of the theorems above show there really does exist a determinant function \( D: \text{Mat}_n(\mathbb{R}) \to \mathbb{R} \) which is alternating, multilinear, and normalized. We'll handle the uniqueness later.

Prop. If \( A \) is an upper triangular or lower triangular matrix, then \( \det(A) = a_{11}a_{22} \cdots a_{nn} \) is the product of the entries along the main diagonal.

Pf. Let's do upper triangular; the proof in the lower triangular case is similar.

So \( A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
0 & 0 & \cdots & a_{nn}
\end{bmatrix} \)
We prove this by induction.

When \( n = 1 \), \( A = [a_{11}] \), \( \det A = a_{11} \) and the result is obvious. Now assume the result is true for all \((n-1) \times (n-1)\) matrices. Expanding by minors along 1st column gives

\[
\det A = a_{11} c_{11} = a_{11} (l)^2 \det \begin{bmatrix}
a_{22} & \cdots & a_{2n} \\
0 & \cdots & 0 \\
\end{bmatrix},
\]

but this minor is an upper triangular \((n-1) \times (n-1)\) matrix, so by the induction hypothesis,

\[
\det \begin{bmatrix}
a_{22} & \cdots & a_{2n} \\
0 & \cdots & 0 \\
\end{bmatrix} = a_{22} a_{23} \cdots a_{2n}.
\]

Thus \( \det A = a_{11} a_{22} \cdots a_{nn} \). \( \Box \)

We used

\textit{Induction principle}. If \( P(n) \) is a statement depending on an integer \( n \) and \( P(1) \) is true, and you can prove that if \( P(k) \) is true then so is \( P(k+1) \) for any \( k \geq 1 \), then \( P(n) \) is true for all \( n \geq 1 \).
Here is another example of proof by induction.

Thm. \[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \] for all \( n \geq 1 \).

Pf. when \( n=1 \), \[ \frac{1}{i=1} \sum i = 1 = \frac{1(1+1)}{2} \] holds.

If \[ \frac{k}{i=1} \sum i = \frac{k(k+1)}{2} \] is true, then

\[ \frac{k+1}{i=1} \sum i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) \]

\[ = \frac{k^2 + k + 2k + 2}{2} = \frac{(k+1)(k+2)}{2} \]

This proves the induction step. So

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \] holds for all \( n \geq 1 \) by induction.