More on determinants.

What are the determinants of the elementary matrices?

if $E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & \cdots & 1 & \\ & & \vdots & \ddots & \vdots & \\ & & 1 & \cdots & 0 & \\ & & & & & \ddots \;\; 1 \end{bmatrix}$ then $\det(E) = \det(I_n E) = -1$ since this is I_n with 2 columns switched.

if $E = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & 1 & \\ & & & & \ddots \;\; 1 \end{bmatrix}$ then $\det(E) = c$ since this is I_n with one column multiplied by c.

if $E = \begin{array}{c} \begin{matrix} & & & & & i & j & & \end{matrix} \\ \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & c & \ddots & \\ & & & & \ddots \;\; 1 \end{bmatrix} \end{array}$ then $\det(E) = 1$ since this is I_n with c times jth column added to ith column.

Now notice: if E is an elementary matrix, then E^T is also an elementary matrix of the same type and so $\det(E) = \det(E^T)$.

As a corollary we get

Thm If A is an $n\times n$ square matrix, then $\det(A) = \det(A^T)$.

Pf. If A is singular, i.e. $\operatorname{rank}(A) < n$, then $\det(A) = 0$. But then $\operatorname{rank}(A^T) = \operatorname{rank}(A) < n$, so $\det(A^T) = 0$ as well.

Now assume A is nonsingular. Then A has row echelon form I_n, so $EA = I_n$ where $E = E_1 E_2 \cdots E_k$ is a product of elementary matrices E_i. Then $\det(EA) = \det(I_n) = 1$

$= \det(E)\det(A)$

while if we take the transpose, we get

$$A^T E^T = (I_n)^T = I_n \quad \text{so}$$

$$\det(A^T)\det(E^T) = 1.$$

Since $E^T = (E_k)^T (E_{k-1})^T \cdots (E_1)^T$

we have $\det(E^T)$

$$= \det(E_k^T) \cdots \det(E_1^T)$$
$$= \det(E_k) \cdots \det(E_1)$$
$$= \det(E_1 \cdots E_k) = \det(E).$$

(since $\det(E_i) = \det(E_i^T)$ for an elementary matrix). So $\det(A) = \det(A^T)$. D.

We still haven't shown that there really is a determinant function that is alternating, multilinear, and normalized, and unique with ~~they~~ these properties.

Def. Let A be an $n \times n$ matrix with $n \geq 2$. Given i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$ we define the (i,j)th cofactor of A to be

$c_{ij} = (-1)^{i+j} \cdot$ the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing row i and column j of A.

(Such an $(n-1) \times (n-1)$ matrix is called a minor of A).

Ex. $A = \begin{bmatrix} 1 & 4 & -1 \\ 5 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ $\quad c_{3,2} = (-1)^5 \det \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}$

$= -7.$

Thm (expansion by minors along a row)

Let $n \geq 2$ and fix one i with $1 \leq i \leq n$.

The function $D: M_{n \times n}(\mathbb{R}) \longrightarrow \mathbb{R}$

$$A \longmapsto \sum_{j=1}^{n} a_{ij} c_{ij}$$

is alternating, multilinear, and normalized and so is the determinant function.

Ex. $A = \begin{bmatrix} 1 & 2 & 5 & 4 \\ 0 & 1 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

We calulate $\det(A)$ by expansion by minors along row 2 (rows with 0's make things easier.)

$$\det(A) = \sum_{j=1}^{4} a_{2j} c_{2j}$$

$$= 0(-1)^{2+1} \begin{vmatrix} 2 & 5 & 4 \\ 4 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} + 1(-1)^{2+2} \begin{vmatrix} 1 & 5 & 4 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix}$$

$$+ (-1)(-1)^{2+3} \begin{vmatrix} 1 & 2 & 4 \\ 3 & 4 & 0 \\ 1 & 1 & -1 \end{vmatrix} + 0(-1)^{2+4} \begin{vmatrix} * \end{vmatrix}$$

doesn't matter

$$= \begin{vmatrix} 1 & 5 & 4 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 4 \\ 3 & 4 & 0 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 11 - (4-15) + (-4+12) - (16-6)$$

$$= 11 - (-11) + (8) - (10)$$

$$= 20.$$

We can expand by columns just as well.

<u>Thm.</u> (expansion by minors along a column)
Let $n \geq 2$ and j be fixed.
The function $D: M_{n \times n}(\mathbb{R}) \longrightarrow \mathbb{R}$

$$A \longmapsto \sum_{i=1}^{n} a_{ij} c_{ij}$$

is alternating, multilinear, and normalized and so is the determinant.

These proofs are a little technical so I won't do the proofs in class. See the text for a proof of expansion by minors along a row.
The column version follows by taking transposes and using $\det(A) = \det(A^T)$.

Ex. $A = \begin{bmatrix} 1 & 2 & 5 & 4 \\ 0 & 1 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

find $\det(A)$ by expanding along 4th column.

$$-4 \begin{vmatrix} 0 & 1 & -1 \\ 3 & 4 & 1 \\ 1 & 1 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 2 & 5 \\ 0 & 1 & -1 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= -4(-1) \quad + (-1)(-16) = 20.$$

Either of the theorems above shows there really does exist a determinant function $D: M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ which is alternating, multilinear, and normalized. We'll handle the uniqueness later.

Prop. If A is an upper triangular or lower triangular matrix, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$ is the product of the entries along the main diagonal.

Pf. Let's do upper triangular; the proof in the lower triangular case is similar.

So $A = \begin{bmatrix} a_{11} & & & * \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$

We prove this by induction.

When $n=1$, $A = [a_{11}]$, $\det A = a_{11}$ and the result is obvious. Now assume the result is true for all $(n-1)\times(n-1)$ matrices. Expanding by minors along 1st column gives

$$\det A = a_{11} C_{11} = a_{11} (-1)^2 \det \begin{bmatrix} a_{22} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}.$$

but this minor is an upper triangular $(n-1)\times(n-1)$ matrix, so by the induction hypothesis,

$$\det \begin{bmatrix} a_{22} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix} = a_{22} a_{33} \cdots a_{nn}.$$

thus $\det A = a_{11} a_{22} \cdots a_{nn}$. $\square$.

We used

Induction principle. If $P(n)$ is a statement depending on an integer n and $P(1)$ is true, and you can prove that if $P(k)$ is true then so is $P(k+1)$ for any $k \geq 1$, then $P(n)$ is true for all $n \geq 1$.

Here is another example of proof by induction.

<u>Thm.</u> $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \geq 1$.

<u>Pf.</u> when $n=1$, $\displaystyle\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$ holds.

If $\displaystyle\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ is true, then

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + k+1 = \frac{k(k+1)}{2} + k+1$$

$$= \cancel{k+k} \ \frac{k^2 + k + 2k + 2}{2}$$

$$= \frac{(k+1)(k+2)}{2}.$$

This prove the induction step. So

$\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ holds for all $n \geq 1$

by induction.