Solutions to Selected Exercises from Homework 5

5.1.12 Suppose for the sake of contradiction that there were no such $N \in \mathbb{N}$ with the property that $X \subseteq \cup_{i=1}^{N} U_i$. Define a sequence $\{x_k\} \subseteq X$ with the property that $x_k \notin \cup_{i=1}^{k} U_i$. By compactness of X, there must be some subsequence $\{x_{k_j}\}$ which converges to some point $b \in X$. As $X \subset \cup_{i=1}^{\infty} U_i$, we deduce that $b \in \cup_{i=1}^{\infty} U_i$. In other words, there is some M such that $b \in U_M$. By virtue of U_M being open, we know there exists some $\varepsilon > 0$ such that $B(b, \varepsilon) \subset U_M$. By construction, we also know there exists some $J > 0$ such that for $j > J$, $x_{k_j} \in B(b, \varepsilon)$. But for $k_j > M$, this contradicts the choice that $x_{k_j} \notin \cup_{i=1}^{k_j} U_i$. Therefore, it must be the case that there is some $N \in \mathbb{N}$ with the property that $X \subset \cup_{i=1}^{N} U_i$.

5.2.3 Let

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

denote a vertex of the rectangular box. Of course, in order to maximize volume it must be the case that $\vec{x}$ is on the hemisphere $x^2 + y^2 + z^2 = r^2$. Using the relation $z = \sqrt{r^2 - x^2 - y^2}$, we define the volume function

$$\begin{aligned} V : D &\to \mathbb{R} \\ \begin{bmatrix} x \\ y \end{bmatrix} &\mapsto (2x)(2y)\sqrt{r^2 - x^2 - y^2} \end{aligned}$$

where we define the domain

$$D := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x, y \geq 0,\ r^2 - x^2 - y^2 \geq 0 \right\}.$$

We note that V is continuous and D is compact, so the Maximum Value Theorem applies. We first hunt for critical points. Observe that

$$DV(x, y) = \frac{4}{\sqrt{r^2 - x^2 - y^2}} \begin{bmatrix} y(r^2 - 2x^2 - y^2) & x(r^2 - x^2 - 2y^2) \end{bmatrix}.$$

In order for this to be the zero matrix, either $x = y = 0$, or we have the system of quadratic equations

$$\begin{aligned} r^2 - 2x^2 - y^2 &= 0 \\ r^2 - x^2 - 2y^2 &= 0. \end{aligned}$$

In the former case, we see that $V(x, y) = 0$ which is clearly a minimum. In the latter case, we find that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{r}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

At the above mentioned point, we have $V(x, y) = \frac{4r^3}{3\sqrt{3}}$. We now check the boundary to ascertain whether or not this is indeed the global maximum. It's clear that if $x = 0, r$ or $y = 0, r$ then the volume is zero. Likewise, if $x^2 + y^2 = r^2$, then the volume is also zero. Hence, the maximal volume is $\frac{4r^3}{3\sqrt{3}}$.