5.1.12 Suppose for the sake of contradiction that there were no such \( N \in \mathbb{N} \) with the property that \( X \subseteq \bigcup_{i=1}^{N} U_i \). Define a sequence \( \{x_k\} \subseteq X \) with the property that \( x_k \notin \bigcup_{i=1}^{k} U_i \). By compactness of \( X \), there must be some subsequence \( \{x_{k_j}\} \) which converges to some point \( b \in X \). As \( X \subseteq \bigcup_{i=1}^{\infty} U_i \), we know there must be some subsequence \( \{x_{k_j}\} \) which converges to some point \( b \in X \). By virtue of \( X \) being compact, we deduce that \( b \in \bigcup_{i=1}^{\infty} U_i \). In other words, there is some \( M \) such that \( b \in U_M \). By construction, we also know there exists some \( \varepsilon > 0 \) such that \( B(b, \varepsilon) \subseteq U_M \). By construction, we also know there exists some \( J > 0 \) such that for \( j > J \), \( x_{k_j} \in B(b, \varepsilon) \). But for \( k_j > M \), this contradicts the choice that \( x_{k_j} \notin \bigcup_{i=1}^{k_j} U_i \). Therefore, it must be the case that there is some \( N \in \mathbb{N} \) with the property that \( X \subseteq \bigcup_{i=1}^{N} U_i \).

5.2.3 Let \( \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) denote a vertex of the rectangular box. Of course, in order to maximize volume it must be the case that \( \vec{x} \) is on the hemisphere \( x^2 + y^2 + z^2 = r^2 \). Using the relation \( z = \sqrt{r^2 - x^2 - y^2} \), we define the volume function

\[
V : D \to \mathbb{R} \\
\begin{bmatrix} x \\ y \end{bmatrix} \mapsto (2x)(2y)\sqrt{r^2 - x^2 - y^2}
\]

where we define the domain

\[
D := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x, y \geq 0, \ r^2 - x^2 - y^2 \geq 0 \right\}.
\]

We note that \( V \) is continuous and \( D \) is compact, so the Maximum Value Theorem applies. We first hunt for critical points. Observe that

\[
DV(x, y) = \frac{4}{\sqrt{r^2 - x^2 - y^2}} \begin{bmatrix} y(r^2 - 2x^2 - y^2) & x(r^2 - x^2 - 2y^2) \end{bmatrix}.
\]

In order for this to be the zero matrix, either \( x = y = 0 \), or we have the system of quadratic equations

\[
\begin{align*}
    r^2 - 2x^2 - y^2 &= 0 \\
    r^2 - x^2 - 2y^2 &= 0.
\end{align*}
\]

In the former case, we see that \( V(x, y) = 0 \) which is clearly a minimum. In the latter case, we find that

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \frac{r}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

At the above mentioned point, we have \( V(x, y) = \frac{4r^3}{3\sqrt{3}} \). We now check the boundary to ascertain whether or not this is indeed the global maximum. It’s clear that if \( x = 0, r \) or \( y = 0, r \) then the volume is zero. Likewise, if \( x^2 + y^2 = r^2 \), then the volume is also zero. Hence, the maximal volume is \( \frac{4r^3}{3\sqrt{3}} \).