MATH 31BH WINTER 2018 MIDTERM 2 SOLUTIONS

1 (15 pts). (Show your work, but no proof required). Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f\begin{pmatrix} x \\ y \end{pmatrix} = 3x^2 + xy^2 - 2y^3$. Let $\vec{a} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

(a) (5 pts). Compute the gradient of f at $\vec{a}$.

(b) (5 pts). Compute the directional derivative of f at $\vec{a}$ in the direction $\vec{v}$.

(c) (5 pts). For what value c does the point $\vec{a}$ lie on the level curve $f\begin{pmatrix} x \\ y \end{pmatrix} = c$? For that value of c, give an equation of the the tangent line to the level curve $f\begin{pmatrix} x \\ y \end{pmatrix} = c$ at the point $\vec{a}$.

Solution.

(a). We have $\partial f/\partial x = 6x + y^2$ and $\partial f/\partial y = 2xy - 6y^2$. Thus $\nabla f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 6x + y^2 \\ 2xy - 6y^2 \end{bmatrix}$ and so $\nabla f(\vec{a}) = \begin{bmatrix} -11 \\ -10 \end{bmatrix}$.

(b). We use that $D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$. In this case we get $\begin{bmatrix} -11 \\ -10 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 22 - 20 = 2$.

(c). $c = f(\vec{a}) = 12 - 2 - 2 = 8$. Then we use that the gradient is orthogonal to the level curve, and hence also to the tangent line to the level curve, at that point. So the tangent line is $\{\vec{x} \,|\, (\vec{x} - \vec{a}) \cdot \nabla f(\vec{a}) = 0\}$, or $-11(x + 2) - 10(y - 1) = 0$.

2 (15 pts). Let A be an $n \times n$ matrix. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = A\vec{x} \cdot \vec{x} = \vec{x}^T A\vec{x}$.

(a) (10 pts). Prove using the definition of the derivative that f is differentiable and that

$$Df(\vec{a})\vec{h} = A\vec{a} \cdot \vec{h} + A\vec{h} \cdot \vec{a}.$$

(b) (5 pts). Fix some vector $\vec{v} \in \mathbb{R}^n$ and let $g : \mathbb{R} \to \mathbb{R}^n$ be given by $g(t) = t^2\vec{v}$. Show that $f \circ g : \mathbb{R} \to \mathbb{R}$ is differentiable at each t and find a formula for $(f \circ g)'(t)$.

Solution.

(a). Since we are given the formula for the derivative, to prove differentiability we just need to check that

$$\lim_{\vec{h} \to 0} \frac{f(\vec{a} + \vec{h}) - f(\vec{a}) - Df(\vec{a})\vec{h}}{\|h\|} = \lim_{\vec{h} \to 0} \frac{A(\vec{a} + \vec{h}) \cdot (\vec{a} + \vec{h}) - A\vec{a} \cdot \vec{a} - A\vec{a} \cdot \vec{h} - A\vec{h} \cdot \vec{a}}{\|h\|} = 0.$$

Expanding, the numerator of our limit is equal to

$$A\vec{a}\cdot\vec{a} + A\vec{h}\cdot\vec{a} + A\vec{a}\cdot\vec{h} + A\vec{h}\cdot\vec{h} - A\vec{a}\cdot\vec{a} - A\vec{a}\cdot\vec{h} - A\vec{h}\cdot\vec{a} = A\vec{h}\cdot\vec{h}.$$

Using Cauchy-Schwarz, we have

$$|A\vec{h}\cdot\vec{h}| \leq \left\|A\vec{h}\right\|\left\|\vec{h}\right\| \quad \text{and thus} \quad \left|\frac{A\vec{h}\cdot\vec{h}}{\left\|\vec{h}\right\|}\right| \leq \left\|A\vec{h}\right\|.$$

Since the function $\mathbb{R}^n \to \mathbb{R}$ given by $\vec{x} \to A\vec{x}$ is linear, it is continuous, so we have $\lim_{\vec{h}\to 0}\left\|A\vec{h}\right\| = A\vec{0} = 0$. Hence

$$\lim_{\vec{h}\to 0}\frac{A\vec{h}\cdot\vec{h}}{\left\|\vec{h}\right\|} = 0$$

as well by the squeeze theorem.

(b). We just saw that f is differentiable at each $\vec{a}$. The coordinate functions of g are all polynomials in t and hence g is differentiable at every t as well. Thus by the chain rule $f\circ g$ will be differentiable at each t, and we will have $(f\circ g)'(t) = Df(g(t))\circ Dg(t)$.

We calculated Df above and so $Df(g(t))$ is the linear transformation with

$$Df(g(t))(\vec{h}) = Df(t^2\vec{v})(\vec{h}) = At^2\vec{v}\cdot\vec{h} + A\vec{h}\cdot t^2\vec{v} = t^2(A\vec{v}\cdot\vec{h} + A\vec{h}\cdot\vec{v}).$$

Since the ith coordinate function of g is $g_i = t^2v_i$, we have $dg_i/dt = 2tv_i$ and hence the Jacobian matrix of g is $\begin{bmatrix} 2tv_1 \\ \dots \\ 2tv_n \end{bmatrix}$, so that $Dg(t): \mathbb{R} \to \mathbb{R}^n$ is given by

$$[Dg(t)](u) = u\begin{bmatrix} 2tv_1 \\ \dots \\ 2tv_n \end{bmatrix} = 2tu\vec{v}.$$

Then

$$[Df(g(t))\circ Dg(t)](u) = Df(g(t))(2ut\vec{v}) = t^2(A\vec{v}\cdot 2ut\vec{v} + A(2ut\vec{v})\cdot\vec{v}) = 4t^3(A\vec{v}\cdot\vec{v})u$$

and so $(f\circ g)'(t) = 4t^3(A\vec{v}\cdot\vec{v})$.

3 (15 pts). Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} y^3/x & x \neq 0 \\ 0 & x = 0 \end{cases}$.

(a) (5 pts). Show that f has directional derivatives equal to 0 in every direction at the origin $\vec{0}$.

(b) (5 pts). Show that f is unbounded in every neighborhood of $\vec{0}$.

(c) (5 pts). Is f differentiable at $\vec{0}$?

Solution.

(a). Consider $D_{\vec{v}}f$. If $v_1 = 0$, then $f(t\vec{v}) = 0$ for all t by definition, so

$$D_{\vec{v}}f = \lim_{t\to 0}(f(t\vec{v}) - f(\vec{0}))/t = \lim_{t\to 0}(0-0)/t = 0.$$

On the other hand, if $v_1 \neq 0$, then

$$D_{\vec{v}}f = \lim_{t\to 0}(f(t\vec{v}) - f(\vec{0})/t = \lim_{t\to 0}(tv_2)^3/t(tv_1) = \lim_{t\to 0}(v_2^3/v_1)t = 0.$$

To justify the last equality we can note that $(v_2^3/v_1)t$ is a polynomial function of t and hence is continuous.

(b). Let $\delta > 0$ and consider the neighborhood $B(\vec{0}, \delta)$. Fix $M > 0$. We need to show that there is a point $\vec{x} \in B(\vec{0}, \delta)$ such that $f(\vec{x}) > M$. Choose a number t such that $0 < t < \min(1/M, 1, \delta/\sqrt{2})$. Consider the point $\vec{x} = \begin{bmatrix} t^4 \\ t \end{bmatrix}$. Since $t < 1$, we have $t^8 < t^2$. Thus $\|\vec{x}\| = \sqrt{t^8 + t^2} < \sqrt{t^2 + t^2} = \sqrt{2}t < \delta$ and so $\vec{x} \in B(\vec{0}, \delta)$. On the other hand, $f(\vec{x}) = t^3/t^4 = 1/t > M$.

(c). In fact f is not even continuous at $\vec{0}$. Then by a theorem we proved, f cannot be differentiable at $\vec{0}$.

One way to see that f is not continuous is to use part (b). If f is continuous at $\vec{0}$, then given any $M > 0$, there is a $\delta > 0$ such that $|f(\vec{x})| < M$ for all $\vec{x}$ with $\|\vec{x}\| < \delta$. But we've just shown in (b) that for any $M > 0$, no matter what $\delta > 0$ we pick there is an $\vec{x}$ that violates this, so this is a contradiction.

One can also show discontinuity directly. Here is a proof involving sequences: Define $\vec{x_k} = \begin{bmatrix} 1/k^3 \\ 1/k \end{bmatrix}$ for each natural number k. Then $f(\vec{x_k}) = 1$ for all k. The limits $\lim_{k\to\infty} 1/k = 0$ and $\lim_{k\to\infty} 1/k^3 = 0$ are standard, and so $\lim_{k\to\infty} \vec{x_k} = \vec{0}$ because the limit of a sequence can be computed in each coordinate. Then if f is continuous, we must have $\lim_{k\to\infty} f(\vec{x_k}) = f(\vec{0}) = 0$. But $\lim_{k\to\infty} f(\vec{x_k}) = \lim_{k\to\infty} 1 = 1$, a contradiction.