MATH 20C Practice Final solutions

Problem 1. Two vectors are orthogonal if and only if their dot product yields 0. So,

$$\begin{aligned}(7, x, -10) \cdot (3, x, x) &= 0 \\ 21 + x^2 - 10x &= 0 \\ (x-5)^2 - 4 &= 0 \qquad \text{(By completing the square)} \\ x &= 3 \text{ or } 7\end{aligned}$$

Problem 2. Let the three points be A $= (0, -2, -1)$, B $= (1, 4, 0)$, and C $= (2, 10, 1)$. If they are colinear, it must be the case that the vectors connecting any pair of points out of the three are parallel. The converse is also true. In particular, the vector from A to B is $(1, 6, 1)$ and the vector from B to C is $(1, 6, 1)$. These two vectors are parallel, so A, B, and C are colinear.

Note that another way to check for parallel vectors is to use cross product. The cross product of two parallel vectors is 0.

Problem 3. To define an equation for the plane, we need a normal vector and a point on the plane. Because the given line $l(t) = (0, -2, 1) + t(1, -2, 3)$ is perpendicular to the plane, the vector $(1, -2, 3)$ can be used as a normal vector for the plane. We also know that $(1, 2, -3)$ is a point on the plane, so an equation of the desired plane is

$$\begin{aligned}(x-1) - 2(y-2) + 3(z+3) &= 0 \\ x - 2y + 3z &= -12\end{aligned}$$

Problem 4. You may use the formula provided in lecture for this problem. The desired distance is

$$\frac{|12(1) + 13(1) - 5(5) + 2|}{\sqrt{12^2 + 13^2 + (-5)^2}} = \frac{2}{\sqrt{(12^2 + 5^2) + 13^2}} = \frac{2}{\sqrt{2 \cdot 13^2}} = \frac{2}{13\sqrt{2}} = \frac{\sqrt{2}}{13}$$

Problem 5.

$$\begin{aligned}\begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 2 \\ 3 & 0 & 1 \end{vmatrix} &= 2\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 4 & 3 \\ 3 & 0 \end{vmatrix} \\ &= 2(3) + (-2) + 0 \\ &= 4\end{aligned}$$

Problem 6. Note that because f is from $\mathbb{R}^2$ to $\mathbb{R}^2$, the derivative of f should be a matrix with two rows and two columns.

$$Df(x, y) = \begin{bmatrix} e^x & 0 \\ y\cos(xy) & x\cos(xy) \end{bmatrix}$$

Problem 7. Because $f(x,y) = e^{xy}$, $\frac{\partial f}{\partial x} = ye^{xy}$, and $\frac{\partial f}{\partial y} = xe^{xy}$. At (0,1), $\frac{\partial f}{\partial x} = 1$ and $\frac{\partial f}{\partial y} = 0$, so the linear approximation to f at (0,1) is given by:

$$f(x,y) \approx f(0,1) + \frac{\partial f}{\partial x}(x-0) + \frac{\partial f}{\partial y}(y-1)$$
$$f(x,y) \approx 1 + x$$

Note: the above means to approximate the value of the function $f(x,y) = e^{xy}$ at a point (x,y) <u>nearby (0,1)</u>, we may approximate it to be $1 + x$.

Problem 8. We have $f(x,y,z) = x^2 + y^2 - z^2$, so $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, and $\frac{\partial f}{\partial z} = -2z$. So,

$$\nabla f(x,y,z) = (2x, 2y, -2z)$$
$$\nabla f(0,0,1) = (0,0,-2)$$

Problem 9. To obtain a parametric equation for a line, we need a point on the line and a direction vector. The point is $\vec{c}(0)$, and $\vec{c'}(0)$ gives us a direction vector for the tangent line. We have,

$$\vec{c}(0) = (1,1) \text{ and}$$
$$\vec{c'}(0) = (1,0)$$

So, the tangent line to $\vec{c}(t)$ at $t = 0$ is given by

$$\vec{l}(t) = (1,1) + t(1,0)$$

Problem 10. The given particle follows the path $\vec{c}(t) = (6t, 3t^2, t^3)$. The velocity of the particle at $t = 1$ is

$$\vec{c'}(1) = (6,6,3)$$

Problem 11. First note that both f and g are functions from $\mathbb{R}^2$ to $\mathbb{R}^2$, their derivatives are both matrices with 2 rows and 2 columns. In particular,

$$Df(u,v) = \begin{bmatrix} sec^2(u-1) & -e^v \\ 2u & -2v \end{bmatrix} \qquad Dg(x,y) = \begin{bmatrix} e^{x-y} & -e^{x-y} \\ 1 & -1 \end{bmatrix}$$

Now,

$$\begin{aligned} D(f \circ g)(0,0) &= Df(g(0,0))Dg(0,0) \\ &= Df(1,0)Dg(0,0) \\ &= \begin{bmatrix} sec^2(1-1) & -e^0 \\ 2(1) & -2(0) \end{bmatrix} \begin{bmatrix} e^{0-0} & -e^{0-0} \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \end{aligned}$$

Problem 12. In this problem, we are looking for the tangent plane to a surface specified as a level set of a (scalar-valued) function from $\mathbb{R}^3$ to $\mathbb{R}$ (the function is $f(x,y,z) = x^3 - 2y^3 + z^3 = 0$), so we can make use of the fact "that the gradient of f at any point (x,y,z) is perpendicular to the level set of f that contains the point (x,y,z)" to deduce that $\nabla f(1,1,1)$ can be used as a normal vector for the desired tangent plane. The gradient of f is

$$\nabla f(x,y,z) = (3x^2, -6y^2, 3z^2)$$

At $(1,1,1)$, the gradient of f is $(3,-6,3)$. So the desired plane has normal vector $(3,-6,3)$ and contains the point $(1,1,1)$ (where it is tangent to the given surface). Thus the equation for the tangent plane is given by

$$\begin{aligned}(3,-6,3)\cdot(x-1,y-1,z-1)&=0\\ 3(x-1)-6(y-1)+3(z-1)&=0\\ 3x-6y+3z&=0\\ x-2y+z&=0\end{aligned}$$

Problem 13. Let $f(x,y)$ be $xe^{1+x^2+y^2}$.The directional derivative of f at $(1,1)$ in the direciton of $(1,-1)$ is $\nabla f(1,1)\cdot(1,-1)$. We have, (don't forget product rule and chain rule from single-variable calculus!)

$$\frac{\partial f}{\partial x}=e^{1+x^2+y^2}+x(e^{1+x^2+y^2}(2x)) \qquad \frac{\partial f}{\partial y}=xe^{1+x^2+y^2}(2y)$$

At (1,1),

$$\frac{\partial f}{\partial x}=e^3+2e^3=3e^3 \qquad \frac{\partial f}{\partial y}=2e^3$$

Thus,

$$\nabla f(1,1)\cdot(1,-1)=(3e^3,2e^3)\cdot(1,-1)=e^3$$

Problem 14. Let $f(x,y)$ be x^2+y^2+3xy. The critical points of f are the points (x,y) at which the derivative of f equals the 0 matrix. That is, they are the points at which

$$Df(x,y)=[2x+3y\ \ 2y+3x]=[0\ \ 0]$$

This lead us to solve the system

$$\begin{cases}2x+3y=0 & (1)\\ 2y+3x=0 & (2)\end{cases}$$

Subtracting equation (1) from (2) yields $x-y=0$, which is to say $x=y$. Substitute this result to (1), we get $5y=0$, which means $y=x=0$. So the only critical point for f is (0,0). To determine whether this point is a local mininum, local maximum, or saddle point, we use the second derivative test. We have

$$\frac{\partial^2 f}{\partial x^2}=2 \qquad \frac{\partial^2 f}{\partial y^2}=2 \qquad \frac{\partial^2 f}{\partial x\partial y}=3 \qquad \text{and}$$
$$(\frac{\partial^2 f}{\partial x^2})(\frac{\partial^2 f}{\partial y^2})-(\frac{\partial^2 f}{\partial x\partial y})^2=-5<0$$

So $(0,0)$ is a saddle point. This means that $(0,0)$ is neither a local min nor a local max. Can you find points at which f is greater than $f(0,0)=0$ and points at which f is less than $f(0,0)=0$?

Problem 15. We proceed by Lagrange Multiplier method. We have, (let $g(x,y)=2x^2+3y^2$)

$$\nabla f(x,y)=(4,2) \qquad \nabla g(x,y)=(4x,6y)$$

The candidate points (x,y) at which $f(x,y)$ is optimized under the constraint $g(x,y)=21$ satisfy

$$\begin{aligned}\nabla f(x,y)&=\lambda\nabla g(x,y)\text{ and }2x^2+3y^2=21\\ (4,2)&=\lambda(4x,6y)\text{ and }2x^2+3y^2=21\end{aligned}$$

which lead us to solve the system

$$\begin{cases} 4 = \lambda 4x & (1) \\ 2 = \lambda 6y & (2) \\ 2x^2 + 3y^2 = 21 & (3) \end{cases}$$

From equation (1) we see that λ could not be 0. We may deduce from equation (1) that $\frac{1}{\lambda} = x$ and from equation (2) that $\frac{1}{\lambda} = 3y$. Hence $x = 3y$. Substitute this into equation (3), we have $18y^2 + 3y^2 = 21$, or $21y^2 = 21$, which has solutions $y = 1$ or $y = -1$. When $y = 1$, $x = 3y = 3$, and when $y = -1$, $x = 3y = -3$. Thus two candidate points for extrema are $(3, 1)$ and $(-3, -1)$. $f(x, y)$ evaluates to 14 and -14 at $(3, 1)$ and $(-3, -1)$ respectively. Note that the set $2x^2 + 3y^2 = 21$ is a closed and bounded set (try drawing it out), and because $f(x, y) = 4x + 2y$ is continuous on this set, by the Extreme Value Theorem, f attains its maximum and minimum on this set. We conclude that 14 and -14 must be the maximum and minimum values of f on $2x^2 + 3y^2 = 21$.

Problem 16. We may separate the ball into its interior ($x^2+y^2+z^2 < 1$) and its boundary ($x^2+y^2+z^2 = 1$, a sphere). Within the interior, we note that $f(x, y, z) = x + y - z$ does not have any critical points because the partial derivatives of f are never 0. But because the ball is a closed and bounded set, and f is continuous over it, by the Extreme Value Theorem, we know that f should attains its (global) maximum and minimum on the ball. Let us try to find extrema of f on the boundary ($x^2 + y^2 + z^2 = 1$) via Lagrange Multipliers.

$$(1, 1 - 1) = \lambda(2x, 2y, 2z) \text{ and } x^2 + y^2 + z^2 = 1$$

$$\begin{cases} 1 = \lambda 2x & (1) \\ 1 = \lambda 2y & (2) \\ -1 = \lambda 2z & (3) \\ x^2 + y^2 + z^2 = 1 & (4) \end{cases}$$

From the first three equations we see that λ is not 0, and $2x = 2y = -2z$ or $x = y = -z$, which implies $x^2 = y^2 = z^2$. By equation (4), this means $x^2 = 1/3 = y^2 = z^2$. The candidate points for extrema are $(\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}})$ and $(-\sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$. At these two points, f evaluates to $3\sqrt{\frac{1}{3}}$ and $-3\sqrt{\frac{1}{3}}$ respectively. We conclude that $3\sqrt{\frac{1}{3}}$ and $-3\sqrt{\frac{1}{3}}$ are the global maximum and global minimum of f on the unit ball.

Problem 17. We are looking for points on the line $x + y = 1$ that optimizes $f(x, y)$. Since the points we are looking for satisfies $y = 1 - x$, we may substitute this into f to obtain a new function in one variable: $g(x) = f(x, 1 - x) = \frac{x^2-(1-x)^2}{x^2+(1-x)^2} = \frac{2x-1}{2x^2-2x+1}$. To find the extrema of a single-variable function, we use the first derivatives to find critical points and use second derivatives to classify local minimum and maximum. We have,

$$\begin{aligned} g(x) &= (2x - 1)(2x^2 - 2x + 1)^{-1} \\ g'(x) &= 2(2x^2 - 2x + 1)^{-1} + (2x - 1)(-1)(2x^2 - 2x + 1)^{-2}(4x - 2) \\ &= 2(2x^2 - 2x + 1)(2x^2 - 2x + 1)^{-2} - 2(2x - 1)^2(2x^2 - 2x + 1)^{-2} \end{aligned}$$

We set $g'(x)$ to 0 to find the critical points:

$$\begin{aligned} g'(x) &= 0 \\ (4x^2-4x+2)(2x^2-2x+1)^{-2} - (8x^2-8x+2)(2x^2-2x+1)^{-2} &= 0 \\ (4x^2-4x+2) - (8x^2-8x+2) &= 0 \\ -4x^2+4x &= 0 \\ 4x(-x+1) &= 0 \\ x &= 1 \text{ or } 0 \end{aligned}$$

Because $g''(1)$ is negative, g has local maximum at $x = 1$; $g''(0)$ is positive, so g has local minimum at $x = 0$. That is, $f(x, y)$ has local minimum when $x = 0$ and $y = 1$ (since $y = 1 - x$) and local maximum when $x = 1$ and $y = 0$. In summary, $f(x, y)$ has local minimum of -1 at (0,1) and local maximum of 1 at (1,0).

Problem 18. Let us separate the constraint into an interior ($x^2 + y^2 < 2$) and a boundary ($x^2 + y^2 = 2$). Note that $\frac{\partial f}{\partial x} = y + 1$ and $\frac{\partial f}{\partial y} = x - 1$ are simultaneously 0 at (1,-1), which does not lie within the interior, so we will not consider (1,-1) as a critical point within the interior (but this point could show up when we solve the Lagrange Multipliers on the boundary; we will consider this point if it occurs then). We now solve for candidate points on the boundary:

$$(y+1, x-1) = \lambda(2x, 2y) \text{ and } x^2 + y^2 = 2$$

$$\begin{cases} y+1 = \lambda 2x & (1) \\ x-1 = \lambda 2y & (2) \\ x^2+y^2 = 2 & (3) \end{cases}$$

If we add equation (1) and (2) together we get $x + y = 2\lambda(x + y)$, or $(x + y)(1 - 2\lambda) = 0$, which has solutions $y = -x$ or $\lambda = \frac{1}{2}$. If $y = -x$, by equation (3), we have (1,-1) and (-1,1) as two solutions. If $\lambda = \frac{1}{2}$, then equation (1) implies $y = x - 1$. Substitute this into equation (3) we obtain $x^2 + (x-1)^2 = 2$, or $2x^2 - 2x - 1 = 0$, which has solutions $x = \frac{1+\sqrt{3}}{2}$ or $x = \frac{1-\sqrt{3}}{2}$. Thus, in the case of $\lambda = \frac{1}{2}$, we have solutions $(\frac{1+\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2})$ and $(\frac{1-\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2})$.

We have 4 candidate points in total, (1,-1), (-1,1), $(\frac{1+\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2})$, and $(\frac{1-\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2})$. Evaluating $f(x, y)$ at each of these points, we get 0, -4, $\frac{1}{2}$, and $-\frac{3}{2}$, respectively. The maximum and minimum values of $f(x, y)$ on the set $x^2 + y^2 \leq 2$ are 1 and -4.

Problem 19. The arc length of the path $\vec{c}(t) = (2t, t^2, ln(t))$ over $1 \leq t \leq 2$ is given by

$$\begin{aligned} &\int_1^2 ||\vec{c'}(t)||dt \\ =&\int_1^2 \sqrt{2^2 + (2t)^2 + (\frac{1}{t})^2}dt \\ =&\int_1^2 \sqrt{(2t + \frac{1}{t})^2}dt \\ =&\int_1^2 (2t + \frac{1}{t})dt \\ =&[t^2 + ln(t)]|_1^2 \\ =&3 + ln(2) \end{aligned}$$

Problem 20.

$$\begin{aligned}\int_{-1}^{0}\int_{1}^{2} -xln(y)dydx &= \int_{-1}^{0} -x\cdot[yln(y)-y]|_1^2 dx \\ &= \int_{-1}^{0} -x(2ln(2)-1)dx \\ &= -(2ln(2)-1)\int_{-1}^{0} xdx \\ &= -(2ln(2)-1)\cdot[\frac{x^2}{2}]|_{-1}^{0} \\ &= \frac{2ln(2)-1}{2}\end{aligned}$$

Problem 21.

$$\begin{aligned}\int_{-2}^{2}\int_{0}^{1} \frac{y}{1+x^2}dxdy &= \int_{0}^{1}\int_{-2}^{2} \frac{y}{1+x^2}dydx \\ &= \int_{0}^{1} 0dx \\ &= 0\end{aligned}$$

Note that we are integrating y, an odd function, over an interval symmetric around 0.

Problem 22. Draw the region to figure outs the bounds of integration!

We may integrate in the order of dydx or dxdy. Below we show the integration dydx.

$$\begin{aligned}\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\int_{0}^{-4y^2+3} x^3ydxdy &= \int_{0}^{3}\int_{-\frac{\sqrt{3-x}}{2}}^{\frac{\sqrt{3-x}}{2}} x^3ydydx \\ &= \int_{0}^{3} x^3\cdot 0dx \\ &= 0\end{aligned}$$

Note that again we are integrating y, an odd function, over an interval symmetric around 0.

Problem 23. Draw the region to figure out how to change the bounds of integration!

$$\begin{aligned}\int_{0}^{1}\int_{\sqrt{y}}^{1} e^{x^3}dxdy &= \int_{0}^{1}\int_{0}^{x^2} e^{x^3}dydx \\ &= \int_{0}^{1} e^{x^3}\int_{0}^{x^2} 1dydx \\ &= \int_{0}^{1} x^2e^{x^3}dx \\ &= \frac{1}{3}\int_{0}^{1} 3x^2e^{x^3}dx \\ &= \frac{1}{3}[e^{x^3}]|_0^1 \\ &= \frac{e-1}{3}\end{aligned}$$