LAST TIME.

Suppose $B$ is a $3 \times 3$ matrix with $\det B \neq 0$ and row vectors $\vec{u}, \vec{v}, \vec{w}$.

1. If $\vec{u}, \vec{v}, \vec{w}$ satisfy the RIGHT HAND RULE then $\det B > 0$.

   OTHERWISE $\det B < 0$.

2. Introduced the cross product of two vectors. If $\vec{v} = (a, b, c)$ and $\vec{w} = (d, e, f)$ then

   \[
   \vec{v} \times \vec{w} = \begin{vmatrix}
   \hat{i} & \hat{j} & \hat{k} \\
   a & b & c \\
   d & e & f \\
   \end{vmatrix} = \hat{i}(bf-ce) - \hat{j}(af-cd) + \hat{k}(ae-bd).
   \]
3. Geometrically...

- \( \mathbf{v} \times \mathbf{w} \) is perpendicular to both \( \mathbf{v} \) & \( \mathbf{w} \).
- \( \mathbf{v} \times \mathbf{w} \) & \( \mathbf{v} \times \mathbf{w} \) satisfy the right hand rule.
- \( ||\mathbf{v} \times \mathbf{w}|| = \text{area of parallelogram} \times \sin \theta \).

Angle between \( \mathbf{v} \times \mathbf{w} \) \( 0 \leq \theta \leq \pi \).

Properties of the Cross Product:

\( \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \)

\( (\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} \)

\( \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} \)

\( (\lambda \mathbf{u}) \times \mathbf{v} = \lambda (\mathbf{u} \times \mathbf{v}) \)

\( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \det \begin{vmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{vmatrix} \)

row 1 = \( \mathbf{u} \)
row 2 = \( \mathbf{v} \)
row 3 = \( \mathbf{w} \)
Example 4. Find a vector perpendicular to the plane \( P \) containing the points \((1,4,2)\), \((0,1,3)\), and \((2,2,5)\).

Take 2 differences & take the cross product.
\[(1,4,2) - (0,1,3) = (1,0,1)\]
\[(2,2,1,5) - (0,1,1,3) = (2,1,2).\]

\[\mathbf{r}(1,0,1) \times (2,1,2) = \int_{0}^{\frac{\pi}{3}} \begin{vmatrix} \mathbf{r} & \mathbf{r} \\ \mathbf{n} & \mathbf{i} \end{vmatrix} \]

\[\mathbf{r}(1,0,1) \times (2,1,2) = (0 \mathbf{i} + 1 \mathbf{j} - 2 \mathbf{k} + (1 - 0) \mathbf{k} = (1, -1, 1).\]
The Equation for a Plane

Start with a point 

\((x_0, y_0, z_0)\)

and a vector \(\vec{n} = (a, b, c)\) perpendicular to \(P\).

Every other point \(Q \in P\) is 

\((x_0, y_0, z_0) + \vec{v}\)

with \(\vec{v}\) perpendicular to \(\vec{n}\).

So: \((x, y, z) \in P\) exactly when:

\[\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0.\]

Equation for \(P\): 

\[a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.\]
Exercise

Find an equation for the plane containing 
\((1,0,0), (0,1,0), (0,0,1)\).

Pf 1. Need a normal vector.
\((0,1,0) - (1,0,0) = (-1,1,0)\)
\((0,0,1) - (0,0,0) = (0,0,1)\).
(2 vectors inside the plane)
\[\begin{align*}
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-1 & 1 & 0 \\
0 & 0 & 1 \\
\end{vmatrix}
= (1,1,1).
\end{align*}\]
Pt 2. Choose a starting point.
   e.g. \( P = (10,0) \)

Equ. for plane:
\[
(1,1,1) \cdot ((x,y,z) - (1,0,0)) = 0
\]
OR \( x - 1 + y + z = 0 \).
OR \( x + y + z = 1 \).
§2.1. GEOMETRY of REAL-VALUED FUNCTIONS.

A function assigns to every 2D (resp. 3D vector) a real number.

Q. How do we visualize these functions?

The **graph** of \( f(x,y) \) is the set of points in \( \mathbb{R}^3 \)

\[ z = f(x,y). \]
Example: \( z = x^2 + y^2 \).

(Q.) How do you graph these?

One method: Level sets.

Choose different values of \( z \), and plot the shape \( f(x,y) = c \) in the \( x-y \) plane.
\( z = 1, 2, 3, 4 \) called level sets of level \( 1, 2, 3, 4 \).

To plot, put the level 1 level set @ height 1, the level 2 level set @ height 2, etc...