HOMEWORK 2 - MATH 100A - DUE FRIDAY OCTOBER 13TH

Problem 1. (From Herstein §2.1 #1) Determine if the following sets G with the operation indicated form a group. If not, point out which of the group axioms fail.

- $G =$ set of all integers $a * b = a - b$.
- $G =$ set of all integers, $A * b = a + b + ab$.
- $G =$ set of nonegative integers, $a * b = a + b$.
- $G =$ set of all rational number $\neq -1$, $a * b = a + b + ab$.
- $G =$ set of all rational numbers with denominator divisible by 5 (written so that numerator and denominator are relatively prime), $a * b = a + b$.
- G a set having more than one element. $a * b = a$ for all $a, b \in G$.

Problem 2. (Herstein §2.1 # 9) If G is a group in which $a^2 = e$ for all $a \in G$, show that G is abelian.

Problem 3. (From Herstein §2.1 # 11) In Example 10, for $n = 3$ find a formula that expresses $(f^i h^j) * (f^s * h^t)$ as $f^a * h^b$. Show that G is a nonabelian group of order 6.

Problem 4. (From Herstein §2.1 # 20) Find all the elements in S_4 such that $x^4 = e$.

Problem 5. (From Herstein §2.1 # 26) If G is a finite group, prove that, given $a \in G$, there is a positive integer n, depending on a such that $a^n = e$.

Problem 6. (From Herstein §2.1 # 27) In the previous problem, show that there is an integer $m > 0$ such that $a^m = e$ for all $a \in G$. (the smallest such integer is sometimes called the *exponent* of a group)

Problem 7. (From Herstein §2.3 #1) If A, B are subgroups of G, show that $A \cap B$ is a subgroup of G.

Problem 8. (From Herstein §2.3 # 7) In S_3 find $C(a)$ for each $a \in S_3$.

Problem 9. (From Herstein §2.3 # 13) If G is cyclic, show that every subgroup of G is cyclic.

Problem 10. Consider the group $G = \mathrm{GL}_2(\mathbb{R})$ of (2×2) invertible matrices with real coefficients. Show that

$$Z(G) = \left\{ A \middle| A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \text{ is diagonal} \right\}.$$

Challenge: Can you prove the analogous statement when $G = \mathrm{GL}_n(\mathbb{R})$ is the group of $(n \times n)$-matrices?