HOMEWORK 3 - MATH 100A - DUE FRIDAY OCTOBER 20TH

Problem 1. (From Herstein §2.4 #5) Let G be a group and H a subgroup of G. Define, for $a, b \in G$, $a \sim b$ if $a^{-1}b \in H$. Prove that this defines an equivalence relation on G, and show that $[a] = aH = \{ah | h \in H\}$. The sets aH are called *left cosets* of H in G.

Problem 2. (Herstein §2.4 # 8) If every right coset of H in G is also a left coset, prove that $aHa^{-1} = H$ for all $a \in G$.

Problem 3. (From Herstein §2.4 # 9) In $\mathbb{Z}_{16}$ write down all the cosets of the subgroup $H = \{[0], [4], [8], [12]\}$. (Since the operation in $\mathbb{Z}_n$ is $+$, write your coset as $[a] + H$. We don't need to distinguish between right cosets and left cosets, sinces $\mathbb{Z}_n$ is abelian under $+$.)

Problem 4. (From Herstein §2.4 # 11) For any finite group G, show that there are as many distinct left cosets of H in G as there are right cosets of H in G.

Problem 5. (From Herstein §2.4 #13) Find the orders of all the elements of U_{18}. Is U_{18} cyclic?

Problem 6. (From Herstein §2.4 #16) If G is a finite abelian group and $a_1, \ldots, a_n$ are all its elements, show that $x = a_1 a_2 \cdots a_n$ must satisfy $x^2 = e$.

Problem 7. (From Herstein §2.4 #21) Let D_8 be the dihedral group of order 8. Find the conjugacy classes in D_8.

Problem 8. (From Herstein §2.4 # 25) Show that the nonzero elements in $\mathbb{Z}_n$ form a group under the product $[a][b] = [ab]$ if and only if n is a prime.

Problem 9. (From Herstein §2.4 # 28) If G is a cyclic group, show that there are $\varphi(n)$ generators for G. Give their form explicitly.

Problem 10. (From Herstein §2.4 # 37 & 38) In a cyclic group of order n, show that for each positive integer m that divides n (including $m = 1$ and $m = n$) there are exactly $\varphi(m)$ elements of order m. Using this result, show that $n = \sum_{m|n} \varphi(m)$.