## HOMEWORK 4 - MATH 100A - DUE FRIDAY OCTOBER 27TH

**Problem 1.** (From Herstein §2.5 #1) Determine in each of the parts if the given mapping is a homomorphism. If so, identify its kernel and whether or not the mapping is 1-1 or onto.

- $G = \mathbb{Z}$ under $+$, $G' = \mathbb{Z}_n$. $\varphi(a) = [a]$ for $a \in \mathbb{Z}$.
- $G$ a group, $\varphi\colon G \to G$ defined by $\varphi(a) = a^{-1}$ for $a \in G$.
- $G$ an abelian group, $\varphi\colon G \to G$ defined by $\varphi(a) = a^{-1}$ for $a \in G$.
- $G$ the group of all nonzero real numbers under multiplication, $G' = \{1, -1\}$, $\varphi(r) = 1$ if $r > 0$ and $\varphi(r) = -1$ if $r$ negative.
- $G$ an abelian group, $n > 1$ a fixed integer, and $\varphi\colon G \to G$ defined by $\phi(a) = a^n$ for all $a \in G$.

**Problem 2.** (Herstein §2.5 # 6) Prove that if $\varphi\colon G \to G'$ is a homomorphism, then $\varphi(G)$ (the image of $G$), is a subgroup of $G'$.

**Problem 3.** (From Herstein §2.5 # 7) Show that $\varphi\colon G \to G'$ is a monomorphism if and only if $\mathrm{Ker}(\varphi) = \{e\}$.

**Problem 4.** (From Herstein §2.5 # 14) If $G$ is abelian and $\varphi\colon G \to G'$ is a surjective homomorphism then $G'$ is abelian.

**Problem 5.** (From Herstein §2.5 #24 parts (a) and (e)) If $G_1$, $G_2$ are two groups, let $G = G_1 \times G_2$ be the Cartesian product of the sets $G_1$ and $G_2$. Define a product in $G$ by $(a_1, b_1) * (a_2, b_2) = (a_1 * a_2, b_1 * b_2)$.

- Prove that $G$ is a group.
- Prove that $G_1 \times G_2 \cong G_2 \times G_1$.

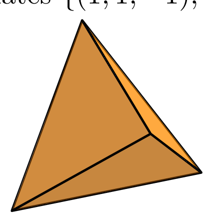
**Problem 6.** (From Herstein §2.5 #26) If $G$ is a group and $a \in G$, define $\sigma_a\colon G \to G$ by $\sigma_a(g) = aga^{-1}$. We saw in Example 9 of this section that $\sigma_a$ is an isomorphism of $G$ onto itself, so $\sigma_a \in A(G)$, the group all 1-1 mappings of $G$ (as a set) onto itself. Define $\psi\colon G \to A(G)$ by $\psi(a) = \sigma_a$ for all $a \in G$. Prove that:

- $\psi$ is a homomorphism of $G$ into $A(G)$.
- $\mathrm{Ker}(\psi) = Z(G)$ (the *center* of G).

**Problem 7.** (From Herstein §2.5 #27) If $\theta$ is an automorphism of $G$ and $N \trianglelefteq G$, prove that $\theta(N) \trianglelefteq G$.

**Problem 8.** Prove that $S_3 \cong D_6$. **Challenge Problem**(=not required): prove that any nonabelian group of order 6 is isomorphic to $S_3$.

**Problem 9.** Let $T \subset \mathbb{R}^3$ denote a *regular tetrahedron* centered at $(0,0,0) \in \mathbb{R}^3$. For example, you can choose the vertices of $T$ to be at the coordinates $\{(1,1,-1), (1,-1,1), (-1,1,1), (-1,-1,-1)\}$.



A tetrahedron.

Define:

$$\mathrm{Isom}(T) = \{A \in \mathrm{GL}_3(\mathbb{R}) | A(T) = T\} = (\text{matrices which give a bijection: } A\colon T \to T) \subset \mathrm{GL}_3(\mathbb{R}).$$

Prove that $\mathrm{Isom}(T)$ is a group, compute its order. (You can use the following fact without proof: any matrix $A$ which maps $T$ bijectively onto $T$ must send the vertices to the vertices.)

**Problem 10.** Prove that $\mathrm{Isom}(T) \cong S_4$.