Math 100A hw3 Sample solution

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1. It suffice to prove that every left coset is also a right coset.
   Since $[G : H] = 2$, then we have exactly 2 left and right cosets of H. Let the set of left cosets be $\{H, gH\}$ and the set of right cosets be $\{H, Hg\}$. Since cosets forms disjoint partition of G, we have $H \bigsqcup gH = G$ which implies $gH = G \backslash H$. For the same reason we have $Hg = G \backslash H$. Hence we have $H = H$ and $gH = G \backslash H = Hg$ and H is normal in G.

2. Since G is cyclic, let $G =< g >$. Let N be a normal subgroup. We will prove that $G/N =< gN >$ which would imply that G/N is cyclic.

   Since $G =< g >$, we have $G = \{e, g^1, g^2, \dots\}$. Hence $G/N = \{N, gN, g^2N, \dots\} =< gN >$.

3. Define $\phi : (R, +) \mapsto (R^{>0}, \times)$ via $\phi(x) = e^x$. We will show that ϕ is isomorphism.

   $\forall\, x, y \in R$, we have $\phi(x + y) = e^{x+y} = e^x e^y = \phi(x)\phi(y)$. Hence ϕ is homomorphism.
   Also, from calculus, we know that e^x is a bijective function on $R \mapsto R^{>0}$. Hence ϕ is isomorphism.

4. Let G be an abelian group and H be a subgroup of G. Then for every $g \in G$, since G is abelian, we have $gHg^{-1} = gg^{-1}H = H$. Hence H is normal in G.

5. Let $x, y \in G$, then $\mathrm{mult}_n(x) + \mathrm{mult}_n(y) = nx + ny = \sum_{i=1}^n x + \sum_{i=1}^n y$. Since G is abelian, we can rearrange the order of the summation. Hence

$$\mathrm{mult}_n(x) + \mathrm{mult}_n(y) = \sum_{i=1}^n x + \sum_{i=1}^n y = \sum_{i=1}^n (x + y) = \mathrm{mult}_n(x + y)$$

   Then we conclude that mult_n is group homomorphism.

   $\ker \mathrm{mult}_n = \{x \in G \mid nx = e\} = \{x \in G \mid \mathrm{order}(x)\big|n\}$

   Some comments: If you think of G as multiplicative group, then nx is just $\prod_{i=1}^n x = x^n$. Then the kernel is the set of elements in G whose order divides n.

6. Let $\sigma, \tau \in S_n$ be disjoint cycles. Then for every $a \in \{1, \dots, n\}$,
   Case 1: $\sigma(a) \neq a$, then a and $\sigma(a)$ are in the cycle of σ. since σ, τ are disjoint cycle, we have $\tau(a) = a$ and $\tau(\sigma(a)) = \sigma(a)$. Hence $\tau\sigma(a) = \tau(\sigma(a)) = \sigma(a)$ and $\sigma\tau(a) = \sigma(a)$.
   Case 2: $\tau(a) \neq a$, follow the same argument as above, we have $\tau\sigma(a) = \sigma\tau(a)$
   Case 3: $\sigma(a) = \tau(a) = a$. Then $\tau\sigma(a) = \tau(a) = a$ and $\sigma\tau(a) = \sigma(a) = a$.
   Hence for all cases, we conclude that $\sigma\tau(a) = \tau\sigma(a)$ for all $a \in \{1, \dots, n\}$. Then we have $\sigma\tau = \tau\sigma$.

7. Let $\sigma \in S_n$ be a k-cycle. Let $\sigma = (a_1, \dots, a_k)$ be the cycle representation of σ. Then observe that $(a_1, \dots, a_k) = (a_k, a_1)(a_{k-1}, a_1) \cdots (a_3, a_1)(a_2, a_1)$ (you can prove this formula by induction on the cycle length k) which is the product of $k-1$ transpositions. Hence σ is an odd permutation when $k-1$ is odd, which is equivalent to k is even.

8. Since conjugation preserve cycle types, hence the set of conjugation classes are exactly cycle types. Here are all of them with representative:

cycle type	representative
(1,1,1,1,1,1)	e
(2,1,1,1,1)	(1,2)
(2,2,1,1)	(1,2)(3,4)
(2,2,2)	(1,2)(3,4)(5,6)
(3,1,1,1)	(1,2,3)
(3,2,1)	(1,2,3)(4,5)
(3,3)	(1,2,3)(4,5,6)
(4,1,1)	(1,2,3,4)
(4,2)	(1,2,3,4)(5,6)
(5,1)	(1,2,3,4,5)
(6)	(1,2,3,4,5,6)

9. $H := \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is a normal subgroup of A_4.
   Proof: Since conjugation preserve the cycle type, and H contains all the permutation of the cycle type (2,2). Hence H is normal in A_4.

10. Let G be finite, simple, and abelian. Suppose $|G|$ is not a prime, say $|G| = pb$ for some prime p and integer $b > 1$. Then by Cauchy's theorem, we can find an element $g \in G$ of order p. Then $< g >$ is a subgroup of G of order p. Since $1 < | < g > | = p < pb = |G|$. Hence $< g >$ is a non-trivial subgroup of G. Further, since G is abelian, by problem 4, we conclude that $< g >$ is normal in G and $< g >$ is non-trivial. This contradicts that G is simple.