HOMEWORK 6 - MATH 100A - DUE FRIDAY NOVEMBER 17TH

Problem 1. Find the center of the group of quaternions, i.e. find $Z(Q_8)$.

Problem 2. Prove that $Q_8/Z(Q_8) \cong K_4$, where K_4 is the Klein 4 group.

Problem 3. Define

$$\text{Heis} = \left\{ A \in \text{GL}_3(\mathbb{R}) \,\middle|\, A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right\} \subset \text{GL}_3(\mathbb{R}).$$

Prove that Heis is a subgroup of $\text{GL}_3(\mathbb{R})$ (called the *Heisenberg group*).

Problem 4. Define

$$N = \left\{ A \in \text{Heis} \,\middle|\, A = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

(1) Prove that N is a normal subgroup of Heis.
(2) Prove that $N \cong \mathbb{R}$, the group of real numbers under addition.

Problem 5. Prove that $\text{Heis}/N \cong \mathbb{R}^2$.

Problem 6. (From Herstein §2.7 #3) Let G be the group of nonzero real numbers under multiplication and let $N = \{1, -1\}$. Prove that $G/N \cong \mathbb{R}^{>0}$, the group of positive real numbers under multiplication.

Problem 7. (From Herstein §2.7 #4) Let G_1 and G_2 be two groups. As in HW4 #5 set

$$G = G_1 \times G_2 = \{(a, b) | a \in G_1, b \in G_2\},$$

where we define $(a, b) * (c, d) = (ac, bd)$, show that

(1) $N = \{(a, e_2) | a \in G_1\}$ where e_2 is the unit element of G_2, is a normal subgroup of G.
(2) $N \cong G_1$.
(3) $G/N \cong G_2$.

Problem 8. (Compare to HW5 #9) Prove that the subset

$$N = \{e, (12)(34), (13)(24), (14)(23)\} \subset A_4$$

is a normal subgroup.

Problem 9. Prove that the quotient $A_4/N \cong \mathbb{Z}_3$.

Problem 10. Prove that every group of order 10 has a normal subgroup of order 5 (**Hint**: use HW5 #1).