Math 100A hw6 Sample solution

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1. By looking at the multiplication table, we have $Z(Q_8) = \{\pm 1\}$

2. $Q_8/Z(Q_8) = \{[1], [i], [j], [k]\}$ and we have the multiplication table:

·	[1]	[i]	[j]	[k]
[1]	[1]	[i]	[j]	[k]
[i]	[i]	[1]	[k]	[j]
[j]	[j]	[k]	[1]	[i]
[k]	[k]	[j]	[i]	[1]

which is the same as the multiplication table of K_4. Hence $Q_8/Z(Q_8) \simeq K_4$.

3. It suffice to prove that Heis is closed under multiplication and inverse.
Let $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & A & B \\ 0 & 1 & C \\ 0 & 0 & 1 \end{bmatrix}$ be two elements of Heis, then

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & A & B \\ 0 & 1 & C \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & A+a & B+aC+b \\ 0 & 1 & C+c \\ 0 & 0 & 1 \end{bmatrix}$$

which is an element in Heis. Hence Heis is closed under multiplication.
The inverse of $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$ which is still an element in Heis. Hence Heis is closed under inverse and we conclude that Heis is a subgroup of $GL_3(\mathbb{R})$

4. (a) We first prove that N is a subgroup of Heis.
Let $\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & B \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be two elements of N, then

$$\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & B \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & B+b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is an element in N. Hence N is closed under multiplication. The inverse of $\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is still an element in N. Hence N is closed under inverse and we conclude that N is a subgroup of Heis.

Next, we will prove that N is normal in Heis.

Let $\begin{bmatrix}1 & 0 & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \in N$ and $\begin{bmatrix}1 & A & B\\ 0 & 1 & C\\ 0 & 0 & 1\end{bmatrix} \in \text{Heis}$, Then

$$\begin{aligned}
\begin{bmatrix}1 & A & B\\ 0 & 1 & C\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & A & B\\ 0 & 1 & C\\ 0 & 0 & 1\end{bmatrix}^{-1} &= \begin{bmatrix}1 & A & B\\ 0 & 1 & C\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -A & AC-B\\ 0 & 1 & -C\\ 0 & 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}1 & A & B+b\\ 0 & 1 & C\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -A & AC-B\\ 0 & 1 & -C\\ 0 & 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}1 & 0 & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \in N
\end{aligned}$$

Hence N is a normal subgroup of Heis.

(b) Consider the map $\phi : N \mapsto \mathbb{R}$ via $\phi\left(\begin{bmatrix}1 & 0 & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = b$. We will prove that this map is an isomorphism.

$$\phi\left(\begin{bmatrix}1 & 0 & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & c\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = \phi\left(\begin{bmatrix}1 & 0 & b+c\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = b+c$$

$$\phi\left(\begin{bmatrix}1 & 0 & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) + \phi\left(\begin{bmatrix}1 & 0 & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = b+c$$

Hence ϕ is group homomorphism.

$\forall x \in \mathbb{R}$, we have $\begin{bmatrix}1 & 0 & x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \in N$ such that $\phi\left(\begin{bmatrix}1 & 0 & x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = x$. Hence ϕ is surjective.

$\begin{bmatrix}1 & 0 & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \in \ker\phi \iff \phi\left(\begin{bmatrix}1 & 0 & b\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\right) = 0 \iff b = 0$. Hence $\ker\phi = \left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right\}$.

Hence ϕ is injective.

Since ϕ is a homomorphism which is surjective and also injective, we conclude that ϕ is isomorphism and $N \simeq \mathbb{R}$

5. Consider that map $\phi : \text{Heis} \mapsto \mathbb{R}^2$ via $\phi\left(\begin{bmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix}\right) = (a, c)$. We will prove that this map is a surjective homomorphism with kernel being N.

$$\phi\left(\begin{bmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & A & B\\ 0 & 1 & C\\ 0 & 0 & 1\end{bmatrix}\right) = \phi\left(\begin{bmatrix}1 & A+a & B+aC+b\\ 0 & 1 & C+c\\ 0 & 0 & 1\end{bmatrix}\right) = (A+a, C+c)$$

$$\phi\left(\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}\right)+\phi\left(\begin{bmatrix}1 & A & B\\0 & 1 & C\\0 & 0 & 1\end{bmatrix}\right)=(a,c)+(A,C)=(A+a,C+c)$$

Hence ϕ is group homomorphism.

Let $(x,y)\in\mathbb{R}^2$, we have $\begin{bmatrix}1 & x & 0\\0 & 1 & y\\0 & 0 & 1\end{bmatrix}\in$ Heis such that $\phi\left(\begin{bmatrix}1 & x & 0\\0 & 1 & y\\0 & 0 & 1\end{bmatrix}\right)=(x,y)$. Hence ϕ is surjective.

If $\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}\in\ker\phi$, then $\phi\left(\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}\right)=(0,0)$ implies that $a=c=0$ and hence $\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & b\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\in N$. Hence $\ker\phi\subset N$.

Let $\begin{bmatrix}1 & 0 & b\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\in N$, then $\phi\left(\begin{bmatrix}1 & 0 & b\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)=(0,0)$ implies that $\begin{bmatrix}1 & 0 & b\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\in\ker\phi$. Hence $N\subset\ker\phi$ and we have $\ker\phi=N$

By first isomorphism theorem, we have $\mathrm{Heis}/N\simeq\mathbb{R}^2$

6. Define $\phi: G\mapsto\mathbb{R}^{>0}$ via $\phi(x)=|x|$. We will prove that this is a surjective homomorphism with kernel being N.

   Let $x,y\in G$, we have $\phi(xy)=|xy|=|x||y|=\phi(x)\phi(y)$. Hence ϕ is group homomorphism.

   Let $x\in R^{>0}$, we can find $x\in G$ such that $\phi(x)=|x|=x$. Hence ϕ is surjective.

   Let $x\in G$, $x\in\ker\phi\Leftrightarrow|x|=1\Leftrightarrow x=\pm1$. Hence $\ker\phi=\{\pm1\}=N$.

   Then by first isomorphism theorem, we have $G/N\simeq\mathbb{R}^{>0}$

7. I will do (3) and (1) first.

   Define $\phi: G\mapsto G_2$ via $\phi\Big((a,b)\Big)=b$.

   $\phi\Big((a_1,b_1)(a_2,b_2)\Big)=\phi\Big((a_1a_2,b_1b_2)\Big)=b_1b_2=\phi\Big((a_1,b_1)\Big)\phi\Big((a_2,b_2)\Big)$. Hence ϕ is group homomorphism.

   For every $b\in G_2$, we have $(e_1,b)\in G$ such that $\phi\Big((e_1,b)\Big)=b$. Hence ϕ is surjective.

   $(a,b)\in\ker\phi\Leftrightarrow\phi\Big((a,b)\Big)=e_2\Leftrightarrow b=e_2\Leftrightarrow(a,b)=(a,e_2)\in N$. Hence $N=\ker\phi$. Hence we conclude that N is a normal subgroup of G and also $G/N\simeq G_2$ by first isomorphism theorem.

   To prove $N\simeq G_1$, it suffice to prove that $\psi: N\mapsto G_1$ via $\psi(a,e_2)=a$ is isomorphism. The proof is similar to previous questions and hence omitted.

8. Observe that N contains all the $(2,2)$ cycle of A_4 and the identity. Conjugating e by other elements will give e which is in N. Since conjugation of permutations will preserve cycle type, hence conjugating a (2,2)-cycle in N by other elements in A_4 will give a (2,2)-cycle which is still an element of N. Hence we conclude that N is normal in A_4

9. By Lagrange, $|A_4/N| = |A_4|/|N| = 12/4 = 3$. Since 3 is prime, hence A_4/N is a cyclic group of order 3, then we have $A_4/N \simeq Z_3$

10. Since $10 = 2 * 5$, then by Cauchy's Theorem, we can find an element $g \in G$ of order 5. Then $< g >$ is a subgroup of index $10/5 = 2$ in G. By hw5 #1, we conclude that $< g >$ is normal in G.