

HOMEWORK 7 - MATH 100A - DUE MONDAY NOVEMBER 27TH

Problem 1. Let G be a group and $H \leq G$. Let S be the set of left cosets of H . Prove that the action of G on S is transitive.

Problem 2. Let $\sigma = (12)(34) \in S_n$ (where $n \geq 4$). Let

$$T_\sigma = [\sigma]$$

denote the conjugacy class of σ . I.e. T_σ consists of permutations of cycle type $2, 2, 1, \dots, 1$. The conjugation action of S_n on T_σ is transitive. Compute the order of the stabilizer, $|\text{Stab}(\sigma)|$. Use this to compute the size of T_σ .

Problem 3. Let $S = \{1, \dots, n\}$ and let

$$\mathcal{P}_k(S) = \{T \subset S \text{ such that } |T| = k\}.$$

An element $\sigma \in S_n$ acts on $T \in \mathcal{P}_k(S)$ by sending:

$$\sigma : T = \{x_1, \dots, x_k\} \mapsto \sigma \cdot T = \{\sigma(x_1), \dots, \sigma(x_k)\}.$$

Prove that the action of S_n on $\mathcal{P}_k(S)$ is transitive.

Problem 4. In the set up of the previous problem, find the stabilizer of $\{1, \dots, k\} \in \mathcal{P}_k(S)$. Conclude that $|\mathcal{P}_k(S)| = \binom{n}{k}$.

Problem 5. Let $\sigma = (12345) \in S_5$. As in Problem 2, let $T_\sigma = [\sigma]$ be the conjugacy class of σ , and consider the action of S_5 on T_σ . Compute $|\text{Stab}(\sigma)|$, and compute $|\text{Stab}(\sigma) \cap A_5|$.

Problem 6. Use your results from Problem 5 to show that not every 5-cycle is conjugate in A_5 .

Problem 7. Let $H \leq G$ be a subgroup of G . Let $g \in G$ and prove that the subset

$$gHg^{-1} = \{ghg^{-1} | h \in H\} \subset G$$

is a subgroup of G . Let $\mathbf{SubGrps}(G)$ be the set of subgroups of G . Prove that the function:

$$G \times \mathbf{SubGrps}(G) \rightarrow \mathbf{SubGrps}(G)$$

$$(g, H) \mapsto g \cdot H := gHg^{-1}$$

is an action of G on $\mathbf{SubGrps}$.

Problem 8. Using the action from Problem 7, prove that a subgroup $H \leq G$ is normal if and only if the orbit of H under the action is a single point, i.e.

$$\text{Orb}(H) = \{H\}.$$

Problem 9. Let p be a prime. Prove that every group of order p^2 has an element of order p .

Problem 10. Consider a regular octagon in \mathbb{C} centered at the origin $0 \in \mathbb{C}$. Let **Paintings** be the set of all possible ways to paint the edges of an octagon with 8 distinct colors. Define an action:

$$D_{16} \times \mathbf{Paintings} \rightarrow \mathbf{Paintings}$$

which permutes the colors of the paintings by symmetries of the octagon. We say two paintings are equivalent if they are in the same orbit. Count the number of orbits, i.e. count the number of paintings up to symmetry.