Math 100A hw7 Sample solution

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1. Pick $eH = H$ be a coset of H. Then $\text{orb}(H) = \{g \cdot H \mid g \in G\} = \{gH \mid g \in G\}$ which equals S, the set of all the cosets of H. Hence the action is transitive.

2. Let $S_n \curvearrowright T_\sigma$ via conjugation. i.e $g \cdot \tau = g\tau g^{-1}$. We will prove that this is a group action first.

   $e \cdot \tau = e\tau e^{-1} = \tau$ for all $\tau \in T_\sigma$
   $g \cdot f \cdot \tau = g \cdot f\tau f^{-1} = gf\tau f^{-1}g^{-1} = (gf)\tau(gf)^{-1} = gf \cdot \tau$ for all $\tau \in T_\sigma$ and $f, g \in S_n$

   Next, pick $\sigma \in T_\sigma$, consider $\text{orb}(\sigma) = \{g \cdot \sigma \mid g \in G\} = \{g\sigma g^{-1} \mid g \in G\} = T_\sigma$. Hence the action is transitive.

   $\text{stab}(\sigma) = \{g \in S_n \mid g\sigma g^{-1} = \sigma\}$. Note that $g\sigma g^{-1} = (g(1), g(2))(g(3), g(4))$. Hence $g\sigma g^{-1} = \sigma$ is equivalent to $\{g(1), g(2)\} = \{1, 2\}; \{g(3), , g(4)\} = (3, 4)$ or $\{g(1), g(2)\} = \{3, 4\}; \{g(3), , g(4)\} = (1, 2)$. Then $|\text{stab}(\sigma)| = 8(n-4)!$. (8 choice for $g(1), g(2), g(3), g(4)$, $(n-4)!$ choice for the other elements). Then by orbit-stabilizer theorem, we have

   $$|T_\sigma| = |\text{orb}(\sigma)| = \frac{|S_n|}{|\text{stab}(\sigma)|} = \frac{n!}{8(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{8}$$

3. Let $N = \{1, \dots, k\}$. We will prove that $\text{orb}(N) = \{\sigma(1), \dots, \sigma(k) \mid \sigma \in S_n\} = \mathcal{P}_k(S)$.

   For every $(x_1, \dots, x_k) \in \mathcal{P}_k(S)$, we can pick $\sigma \in S_n$ such that $\sigma(i) = x_i$ for $0 \leq i \leq k$. Hence $\mathcal{P}_k(S) \subset \text{orb}(N)$. Also, it is clear from the definition of group action and orbits that $\text{orb}(N) \subset \mathcal{P}_k(S)$. Hence $\text{orb}(N) = \mathcal{P}_k(S)$ and the action is transitive.

4. Let $S_n \curvearrowright \{1, \dots, n\}$. The stabilizer of N are permutations in S_n which fixes elements in N. Hence we have $k!$ choice for the first k elements. Then we have $(n-k)!$ choice for rest $(n-k)$ elements. Hence $|\text{stab}(N)| = k!(n-k)!$. Then by the orbit-stabilizer theorem, $|\mathcal{P}_k(S)| = n!/k!(n-k)!$

5. Using same method as in problem 2, we have $|\text{stab}(\sigma)| = 5$ (5 choice for the leading number in the cycle). We can actually write down all the elements in the stabilizer of σ: $\text{stab}(\sigma) = \{e, (12345), (13524), (14253), (15432)\}$. Observe that all those elements are in A_5, hence $|\text{stab}(\sigma) \cap A_5| = 5$

6. Let S be the set of 5-cycles in A_5. We have $|S| = 4! = 24$. Let $A_5 \curvearrowright S$ via conjugation. Using orbit-stabilizer theorem and the previous question, we have $|\text{orb}((12345))| = |A_5|/|\text{stab}((12345))| = 60/5 = 12 \neq 24$. Hence this action is not transitive and we conclude that not every 5-cycle is conjugate in A_5. (However, they are conjugate in S_5)

7. We first prove that gHg^{-1} is a subgroup of G.

   Since $H \leq G$, we have
   $geg^{-1} = e \in gHg^{-1}$
   for every $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$, we have $(gh_1g^{-1})(gh_2g^{-1}) = g(h_1h_2)g^{-1} \in gHg^{-1}$
   for every $ghg^{-1} \in gHg^{-1}$, its inverse $(ghg^{-1})^{-1} = gh^{-1}g^{-1} \in gHg^{-1}$
   Hence gHg^{-1} is a subgroup of G.

   Then we will prove $g \cdot H = gHg^{-1}$ is an action of G on subgroups of G.

   Since gHg^{-1} is a subgroup of G, the function is well-defined.
   $e \cdot H = eHe^{-1} = H$ for all $H \leq G$
   $a \cdot b \cdot H = a \cdot bHb^{-1} = abHb^{-1}a^{-1} = (ab)H(ab)^{-1} = ab \cdot H$ for all $a, b \in G$ and $H \leq G$
   Hence this function is a group action.

8. $H \leq G$ is normal is equivalent to $gHg^{-1} = H$ for all $g \in G$ which is equivalent to $\text{orb}(H) = \{gHg^{-1} \mid g \in G\} = \{H\}$

9. This is actually a direct result of Cauchy's Theorem.
   Here is how to prove it without Cauchy's Theorem: Since $|G| = p^2 \geq 4 > 1$, pick $x \in G - \{e\}$. By Lagrange theorem, order of x is $1, p$ or p^2. Since $x \neq e$, order of x is not 1. If order of x is p, then we are done. Else, suppose the order of x is p^2. Then we have $x^{p^2} = e$ which implies that $(x^p)^p = e$. Since the order of x is p^2, we have $x^p \neq e$, and the order of x^p is not 1. Further, since $(x^p)^p = e$, the order of x^p is a divisor of p, and not 1. Hence the order of x^p is p. For all cases, we can find a element of order p.

10. I will use Burnside Formula to calculate the number of orbits. Let S be the set of all paintings. **If each side can have same colors**, here is the table of fixed points:

$g \in D_{16}$	$\|S^g\|$
e	8^8
r	8
r^2	8^2
r^3	8
r^4	8^4
r^5	8
r^6	8^2
r^7	8
$s, r^2s, r^4s, r^6s, r^8s$	8^4
rs, r^3s, r^5s, r^7s	8^5

   Then by Burnside Formula, the number of orbits $= (8^8 + 8 + 8^2 + \ldots + 8^4)/16 = 1058058$

   Else, **if each side have different colors**, here is the table of fixed points:

$g \in D_{16}$	$\lvert S^g \rvert$
e	$8!$
r	0
r^2	0
r^3	0
r^4	0
r^5	0
r^6	0
r^7	0
$s, r^2s, r^4s, r^6s, r^8s$	0
rs, r^3s, r^5s, r^7s	0

Then by Burnside Formula, the number of orbits $= 8!/16 = 2520$