HOMEWORK 8 - MATH 100A - DUE FRIDAY DECEMBER 1ST

Problem 1. Let $G = \{\pm 1\}$, and let $\mathbb{R}[x]$ be the set of polynomials in 1-variable with $\mathbb{R}$-coefficients. I.e.

$$\mathbb{R}[x] = \{p(x) = a_0 + \cdots + a_d x^d | a_i \in \mathbb{R}\}.$$

Let G act on the set $\mathbb{R}[x]$ by

$$1 \cdot p(x) := p(x), (-1) \cdot p(x) := p(-x).$$

Prove that $\mathrm{Stab}(p(x)) = \{\pm 1\}$ if and only if all the odd coefficients of $p(x)$ are 0 (i.e., if i is odd then $a_i = 0$).

Problem 2. Let G act on a set S. Let $x, y \in S$ be two elements in the same orbit. I.e.

$$x \in \mathrm{Orb}(y).$$

Prove that the subgroups $\mathrm{Stab}(x), \mathrm{Stab}(y) \leq G$ are conjugate. I.e. there exists $g \in G$ such that

$$g^{-1}\mathrm{Stab}(x)g = \mathrm{Stab}(y).$$

Problem 3. Let G be a group which acts on a set S. Let

$$\phi \colon G \to A(S)$$

be the corresponding homomorphism. Prove that

$$\ker(\phi) = \bigcap_{x \in S} \mathrm{Stab}(x).$$

Problem 4. Use the fact that A_5 is simple to prove that there are no nontrivial subgroups $H \leq A_5$ such that $[A_5 : H] < 5$. (Hint: look at the action of A_5 on the left cosets of H.)

Problem 5. Let G be a group which acts on a set S. Let T be an arbitrary set and let $\mathrm{Fun}(S, T)$ be the set of functions from S to T. We define a function

$$G \times \mathrm{Fun}(S, T) \to \mathrm{Fun}(S, T)$$

$$(g, f(x)) \mapsto g \cdot f(x) := f(g^{-1} \cdot x)$$

Prove that this function is a group action.

Problem 6. Define an action of the group $\mathbb{Z}$ on the set of real numbers $\mathbb{R}$ by

$$a \cdot b := 2\pi a + b$$

where $a \in \mathbb{Z}$, and $b \in \mathbb{R}$. Using the action defined in Problem 5, prove that the function $\sin(x) \in \mathrm{Fun}(\mathbb{R}, \mathbb{R})$ has stabilizer

$$\mathrm{Stab}(\sin(x)) = \mathbb{Z}.$$

Problem 7. Using the action in the previous problem. Let $f(x) \in \mathrm{Fun}(\mathbb{R}, \mathbb{R})$. Prove that $\mathrm{Stab}(f(x)) = \mathbb{Z}$ if and only if $f(x)$ is 2π-periodic.

Problem 8. Write down the right hand side of the class equation for the group Q_8. I.e. write

$$8 = 1 + ...$$

where the dots are the sizes of the conjugacy classes.

Problem 9. Write down the right hand side of the class equation for the group D_8.

Problem 10. Write down the right hand side of the class equation for D_{2n}. I.e. figure out the sizes of all the conjugacy classes in D_{2n}. (Hint: consider the cases n is even and n is odd separately.)