1. Let $p(x) = a_0 + \cdots + a_d x^d \in \mathbb{R}[x]$ and suppose $\operatorname{Stab}(p(x)) = \{\pm 1\}$. Note that

$$p(-x) = a_0 + \cdots + a_d(-x)^d = a_0 - a_1 x + a_2 x^2 - \cdots + (-1)^d a_d x^d.$$

Moreover $p(-x) = (-1) \cdot p(x) = p(x)$, because $-1 \in \operatorname{Stab}(p(x))$. Equating coefficients on both sides shows that $a_i = (-1)^i a_i$ for all i, which tells us nothing about the even coefficients, but forces the odd ones to be zero.

Conversely, if $p(x)$ has only even coefficients, then

$$(-1) \cdot p(x) = p(-x) = a_0 + a_2(-x)^2 + \cdots + a_d(-x)^d = a_0 + a_2 x^2 + \cdots + a_d x^d = p(x),$$

so $-1 \in \operatorname{Stab}(p(x))$ and hence $\operatorname{Stab}(p(x)) = \{\pm 1\}$.

2. We know that $x = g \cdot y$ for some $g \in G$. If $s \in \operatorname{Stab}(x)$ then

$$(g^{-1}sg) \cdot y = g^{-1} \cdot (s \cdot (g \cdot y)) = g^{-1} \cdot (s \cdot x) = g^{-1} \cdot x = g^{-1} \cdot (g \cdot y) = (g^{-1}g) \cdot y = y,$$

so $g^{-1}sg \in \operatorname{Stab}(y)$. This shows that $g^{-1} \operatorname{Stab}(x) g \subseteq \operatorname{Stab}(y)$. Since $y = g^{-1} \cdot x$, the same argument shows that $g \operatorname{Stab}(y) g^{-1} \subseteq \operatorname{Stab}(x)$, and hence $\operatorname{Stab}(y) \subseteq g^{-1} \operatorname{Stab}(x) g$. Therefore $\operatorname{Stab}(y) = g^{-1} \operatorname{Stab}(x) g$.

3.

$$\begin{aligned}
\operatorname{Ker}(\varphi) &= \{g \in G \mid \varphi(g) = e_{A(S)}\} \\
&= \{g \in G \mid \varphi(g)(x) = x \text{ for all } x \in S\} \\
&= \{g \in G \mid g \cdot x = x \text{ for all } x \in S\} \\
&= \cap_{x \in S} \{g \in G \mid g \cdot x = x\} \\
&= \cap_{x \in S} \operatorname{Stab}(x).
\end{aligned}$$

4. Let $H \leq A_5$ and let A_5 act on $S := G/H$ by left multiplication. The corresponding homomorphism $\varphi : A_5 \to A(S)$ is either injective or trivial (sending everything to $e_{A(S)}$), because its kernel is either $\{e\}$ or A_5. If it is injective, then

$$60 = |A_5| \leq |A(S)| = |S|! = [G : H]!,$$

so $[G : H] \geq 5$ (for instance $4! = 24$ is too small). Otherwise φ is trivial, so for all $g \in A_5$

$$gH = \varphi(g)(H) = e_{A(S)}(H) = H,$$

which means $g \in H$. In other words $H = A_5$. Therefore any proper subgroup of A_5 has index at least 5.

5. If $f \in \text{Fun}(S,T)$ then $(e \cdot f)(x) = f(e^{-1} \cdot x) = f(e \cdot x) = f(x)$ for all $x \in S$, so $e \cdot f = f$. Moreover, given $g, h \in G$ and $x \in S$

$$\begin{aligned}(g \cdot (h \cdot f))(x) &= (h \cdot f)(g^{-1} \cdot x) \\ &= f(h^{-1} \cdot (g^{-1} \cdot x)) \\ &= f((h^{-1}g^{-1}) \cdot x) \\ &= f((gh)^{-1} \cdot x) \\ &= ((gh) \cdot f)(x),\end{aligned}$$

so $g \cdot (h \cdot f) = (gh) \cdot f$.

6. If $n \in \mathbb{Z}$ then $(n \cdot \sin)(x) = \sin((-n) \cdot x) = \sin(2\pi(-n) + x) = \sin(x)$ for all $x \in \mathbb{R}$, so $n \cdot \sin = \sin$ and $n \in \text{Stab}(\sin)$. Therefore $\text{Stab}(\sin) = \mathbb{Z}$.

7. If $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic, the argument from Problem 6 shows that $\text{Stab}(f) = \mathbb{Z}$. Conversely, if $\text{Stab}(f) = \mathbb{Z}$ then $-1 \in \text{Stab}(f)$, so

$$f(x) = (-1 \cdot f)(x) = f(1 \cdot x) = f(2\pi + x)$$

for all $x \in \mathbb{R}$. In other words f is 2π-periodic.

8. Since 1 and -1 are central in Q_8, they are only conjugate to themselves. Moreover $C(i)$ contains $\langle i \rangle = \{1, i, -1, -i\}$, but not j (as $ij = k$ but $ji = j^2k = -k$), so by Lagrange's theorem $C(i) = \langle i \rangle$. Similarly $C(-i) = \langle i \rangle$. The same argument gives $C(g) = \langle g \rangle$ for all $g \notin C(i)$. By the orbit-stabiliser theorem $|\text{Cl}(g)| = |Q_8|/|C(g)| = 2$ for all $g \neq \pm 1$, so the class equation for Q_8 is $8 = 1 + 1 + 2 + 2 + 2$.

9. See Problem 10.

10. If $k \in \{0, \ldots, n-1\}$ then $r^k s = r^{k-1} s r^{-1} = \cdots = s r^{-k}$, so r^k commutes with s only when $r^k = r^{-k}$, or equivalently $r^{2k} = e$. Since r has order n, this means that n divides $2k$, which only occurs for $k \in \{0, \frac{n}{2}\}$. In these cases $C(r^k)$ contains s and $\langle r \rangle$, so it has at least $n+1$ elements; by Lagrange's theorem $C(r^k) = D_{2n}$. If $k \notin \{0, \frac{n}{2}\}$ then $C(r^k) = \langle r \rangle$ for the same reason. So the conjugacy class of r^k has 1 or 2 elements, depending on whether $k \in \{0, \frac{n}{2}\}$.

    Next, note that $r^{-1}(sr^k)r = (r^{-1}s)r^{k+1} = sr^{k+2}$ and $s^{-1}(sr^k)s = r^k s = sr^{-k}$, so

$$\text{Cl}(sr^k) = \{sr^{k+2i} \mid i \in \mathbb{Z}\}.$$

    If n is odd then this contains sr^i for all $i \in \mathbb{Z}$, so it has n elements, and the class equation is

$$2n = 1 + 2 + \cdots + 2 + n,$$

where 2 appears $\frac{n-1}{2}$ times (there is only one 1 because k cannot be $\frac{n}{2}$). Otherwise $\mathrm{Cl}(sr^k)$ does not contain sr^{k+1}, so $\mathrm{Cl}(s) \neq \mathrm{Cl}(sr)$ and the class equation is

$$2n = 1 + 1 + 2 + \cdots + 2 + \frac{n}{2} + \frac{n}{2},$$

where 2 appears $\frac{n-2}{2}$ times.