Math 103A Homework 3 Solutions

1. We know from Math 109 that $\phi^{-1}$ is a bijection. Let $t_1, t_2 \in T$. We want to show that $\phi^{-1}(t_1 \ast t_2) = \phi^{-1}(t_1) \ast \phi^{-1}(t_2)$. Since $\phi$ is surjective, there exist $s_1, s_2 \in S$ such that $\phi(s_1) = t_1$ and $\phi(s_2) = t_2$ (in fact, $s_1 = \phi^{-1}(t_1)$ and $s_2 = \phi^{-1}(t_2)$). Then

$$\phi^{-1}(t_1 \ast t_2) = \phi^{-1}(\phi(s_1) \ast \phi(s_2)) = \phi^{-1}(\phi(s_1) \ast s_2)$$

since $\phi$ is an isomorphism

$$= s_1 \ast s_2$$

since $\phi^{-1} \circ \phi = id_S$

and

$$\phi^{-1}(t_1) \ast \phi^{-1}(t_2) = \phi^{-1}(\phi(s_1)) \ast \phi^{-1}(\phi(s_2)) = s_1 \ast s_2$$

since $\phi^{-1} \circ \phi = id_S$

Hence, $\phi^{-1}(t_1 \ast t_2) = \phi^{-1}(t_1) \ast \phi^{-1}(t_2)$, so it follows that $\phi^{-1}$ is an isomorphism.

5.(3) We claim that $1 + i$ has infinite order. Suppose, by way of contradiction, that there exists $n \in \mathbb{Z}_+$ such that $(1 + i)^n = 1$. Then

$$1 = |1| = |(1 + i)^n| = |1 + i|^n = (\sqrt{2})^n$$

which is a contradiction since $\sqrt{2} > 1$. Hence, $1 + i$ has infinite order.

5.(4) Direct computation shows that

$$\left(\frac{1 + i}{\sqrt{2}}\right)^1 = \frac{1 + i}{\sqrt{2}}$$

$$\left(\frac{1 + i}{\sqrt{2}}\right)^2 = i$$

$$\left(\frac{1 + i}{\sqrt{2}}\right)^3 = -1 + i$$

$$\left(\frac{1 + i}{\sqrt{2}}\right)^4 = -1$$

$$\left(\frac{1 + i}{\sqrt{2}}\right)^5 = -1 - i$$

$$\left(\frac{1 + i}{\sqrt{2}}\right)^6 = -i$$

$$\left(\frac{1 + i}{\sqrt{2}}\right)^7 = 1 - i$$

$$\left(\frac{1 + i}{\sqrt{2}}\right)^8 = 1$$

Hence $\frac{1 + i}{\sqrt{2}}$ has order 8 in $\mathbb{C}^\ast$. 

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6. Since $G$ is cyclic, there exists $g \in G$ such that $G = \langle g \rangle$. Since $G$ and $H$ are isomorphic, there exists an isomorphism $\phi : G \to H$. We claim that $H = \langle \phi(g) \rangle$. Clearly $\langle \phi(g) \rangle \subseteq H$. Conversely, suppose that $h \in H$. Since $\phi$ is surjective, there exists $g' \in G$ such that $\phi(g') = h$. Since $G = \langle g \rangle$, we know that there exists $n \in \mathbb{Z}$ such that $g' = g^n$. Hence,

$$h = \phi(g') = \phi(g^n) = (\phi(g))^n \in \langle \phi(g) \rangle$$

(the fact that $\phi(g^n) = (\phi(g))^n$ follows from the fact that $\phi(xy) = \phi(x)\phi(y)$ for any $x, y \in G$; if you don’t see why, try to prove it as an exercise). Hence, $H \subseteq \langle \phi(g) \rangle$, so it follows that $H = \langle \phi(g) \rangle$, so $H$ is cyclic.

7.(1) By Theorem 6.14, we know that $\mathbb{Z}_{12}$ has one subgroup of order $d$ for each divisor $d$ of 12. We find that those subgroups are

- $(0) = \{0\}$
- $(6) = \{0, 6\}$
- $(4) = \{0, 4, 8\}$
- $(3) = \{0, 3, 6, 9\}$
- $(2) = \{0, 2, 4, 6, 8, 10\}$
- $(1) = \mathbb{Z}_{12}$

9. Let $x, y \in H \cap K$. Since $H$ is a subgroup and $x, y \in H$, we have that $xy \in H$. Similarly, since $K$ is a subgroup and $x, y \in K$, we have that $xy \in K$. Hence $xy \in H \cap K$, so $H \cap K$ is closed under the group operation.

Now, note that $e \in H$ since $H$ is a subgroup and $e \in K$ since $K$ is a subgroup, so $e \in H \cap K$.

Finally, suppose that $x \in H \cap K$. Since $x \in H$ and $H$ is a subgroup, we have that $x^{-1} \in H$. Similarly, since $x \in K$ and $K$ is a subgroup, we have that $x^{-1} \in K$. Hence, $x^{-1} \in H \cap K$. It follows that $H \cap K$ is a subgroup.