Please take the time to fill out teaching evaluations. Thanks!

**Problem 1.** Show that if $p$ is a prime number, then $|\mathbb{Z}_p^*| = p^k - p^{k-1}$.

**Problem 2.** Is $\mathbb{Z}_9^*$ cyclic? Is $\mathbb{Z}_{15}^*$ cyclic?

**Problem 3.** Write a multiplication table for the group $(\mathbb{Z}_{14}^*)^n$.

**Problem 4.** Let $G_1, \ldots, G_n$ be a sequence of groups. Check that the binary operation
\[
\star : (G_1 \times \cdots \times G_n) \times (G_1 \times \cdots \times G_n) \to (G_1 \times \cdots \times G_n)
\]
defined by $(a_1, \ldots, a_n) \star (b_1, \ldots, b_n) := (a_1 \star b_1, \ldots, a_n \star b_n)$,

makes $G_1 \times \cdots \times G_n$ a group.

**Problem 5.** Let $G_1, \ldots, G_n$ be a sequence of groups. Show that if every $G_i$ is abelian then the direct product $G_1 \times \cdots \times G_n$ is abelian.

**Problem 6.**
- Find the order of $(2, 3)$ in $\mathbb{Z}_6 \times \mathbb{Z}_{15}$.
- Find the order of $(3, 6, 12, 16)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{20} \times \mathbb{Z}_{24}$

**Problem 7.** Let $G$ and $H$ be groups. Define the following projection functions:
\[
p_G : G \times H \to G \text{ defined by } p_G((a, b)) := a,
\]
\[
p_H : G \times H \to H \text{ defined by } p_H((a, b)) := b.
\]
Prove that $p_G$ and $p_H$ are homomorphisms.

**Problem 8.** Using the functions defined in Problem 7, find $\ker(p_G)$ and $\ker(p_H)$.

**Problem 9.** Recall that the Euler totient function, $\varphi$, is defined by
\[
\varphi(n) := |\mathbb{Z}_n^*|.
\]
Prove Euler’s Theorem, which says that if $\gcd(a, n) = 1$ then
\[
a^{\varphi(n)} \equiv 1 \pmod{n}.
\]
(Hint: follow the proof of Fermat’s little theorem.)

**Problem 10.** Show that the group $(\mathbb{Q}, +)$ is not finitely generated, i.e. show that if $S \subset \mathbb{Q}$ is a finite subset, then $\langle S \rangle \neq \mathbb{Q}$.