Math 20E - Final Exam
Due at 10pm, March 18th, 2020

Instructions:

1. You may use the book and your course notes.
2. You may NOT use any other sources. For example:
   (a) You may NOT talk to anyone about the final exam.
   (b) You may NOT use a calculator on the exam.
   (c) You may NOT use the internet when taking the final.
3. You must write your solutions to the exam on paper.
4. You may NOT write your solutions on a tablet.
5. You must submit the exam on gradescope by 10pm on Wednesday.
6. At the top of each page, write your name and student ID number.
7. If you have questions about the exam, email either the professor or one of the TAs.

1. (1 point) Read ALL of the instructions. Write down three reasons why it is wrong to cheat on a final exam.

2. For this problem, let $D \subset \mathbb{R}^2$ be the region of points $(x, y) \in \mathbb{R}^2$ with $1 \leq x^2 + y^2 \leq 4$.

   (a) (1 point) What does D look like? (either draw a picture, or use words.)

   (b) (2 points) Find a parametrization

   $$\Phi\colon R \to D$$

   where $R = [1, 2] \times [0, 2\pi]$.

   (c) (2 points) Use your parametrization from part (b) to evaluate:

   $$\iint_D x^2 dy dx.$$

   (d) (4 points) Let C_1 be the curve $x^2 + y^2 = 1$ and let C_2 be the curve $x^2 + y^2 = 4$ both oriented counterclockwise. Let

   $$\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$$

   be the vector field $\mathbf{F}(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$. Evaluate the integrals:

   $$\int_{C_1} \mathbf{F} \cdot ds \text{ and } \int_{C_2} \mathbf{F} \cdot ds.$$

   (e) (1 point) Let $\mathbf{F}(x, y) = (P(x, y), Q(x, y)) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ be the vector field from the previous problem. Compute:

   $$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

   (f) (1 point) In a sentence or two, explain why your answers in (d) and (e) are consistent with Green's theorem.

3. In this problem, let S be the surface defined be the equations: $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 \leq 1/2$, and $z \geq 0$.

   (a) (1 point) Find a parametrization of S

   $$\Phi\colon D \to \mathbb{R}^3$$

   where $D \subset \mathbb{R}^2$ (Hint: use spherical coordinates).

(b) (2 points) Use part (a) to find the area of S.

(c) (1 point) Let $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F}(x, y, z) = (y, -x, \ln(x^2 + y^2 + 1))$. Compute $\nabla \times \mathbf{F}$.

(d) (3 points) Let $\mathbf{F}$ be the vector field from part (c). Use Stokes' theorem to evaluate

$$\iint_{\Phi} (\nabla \times \mathbf{F}) \cdot dA$$

where Φ is the parametrization from part (a).

4. In this problem, let $V \subset \mathbb{R}^3$ be the region described by the equations:

$$x^2 + y^2 \leq z^2 \text{ and } 0 \leq z \leq 1.$$

(a) (3 points) Compute the volume of V.

(b) (1 point) Let $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F}(x, y, z) = (0, 0, z)$. Evaluate:

$$\iiint_{V} \nabla \cdot \mathbf{F} dz dy dx.$$

(c) (3 points) Let S be the surface defined by the equations $x^2 + y^2 = z^2$ and $0 \leq z \leq 1$, oriented outwards. Use Gauss's theorem to evaluate

$$\iint_{S} (0, 0, z) \cdot dA.$$

(Warning! ∂V consists of S and another piece.)

5. Let $R \subset \mathbb{R}^2$ and $S \subset \mathbb{R}^2$ be two discs of radius 1. R is centered at the point $(0, 0)$ and S is centered at $(1, 1)$. Let D be the set of points contained in both R and S.

(a) (1 point) Draw a picture of R, S, and D.

(b) (2 points) Let C be the boundary of D, oriented counterclockwise. C has two parts. Parametrize both of them.

(c) (1 point) What is the length of C?

(d) (4 points) Evaluate the integral:

$$\frac{1}{2} \int_{C} x dy - y dx.$$

(e) (1 point) Use your answer from part (d) and Green's theorem to compute the area of D.