Instructions:

Do not open the exam until you are instructed to do so.

Write your name and student ID number on the front page of the exam.

Write your name and student ID number at the top of every page of the exam.

Answer the questions in the spaces provided. If you run out of room for an answer, continue on the back of the page.

You must show your work!

There is an extra page at the end of the exam for scratch work.

If you need more scratch paper, ask one of the proctors and we will provide it. This additional paper will not be graded.

Math 20E - Midterm 2 - 2/26/2020

Name & Student ID: _____

Question:	1	2	3	4	Total
Points:	4	4	6	7	21
Score:					

Name & Student ID: _____

1. Know your definitions.

   (a) (1 point) Let C be a curve in $\mathbb{R}^2$ which is parametrized by a one-to-one function:

   $$\mathbf{p}(t)\colon [a,b]\to C.$$

   Let $f(x,y)$ be a scalar function on $\mathbb{R}^2$. Write an integral with respect to t which computes $\int\limits_C f(x,y)ds$.

   $$\int\limits_C f(x,y)ds = \int\limits_a^b f(\mathbf{p}(t))||\mathbf{p}'(t)||ds$$

   (b) (1 point) Let C be a curve in $\mathbb{R}^2$ which is parametrized by a one-to-one function:

   $$\mathbf{p}(t)\colon [a,b]\to C.$$

   Let $\mathbf{V}\colon \mathbb{R}^2\to\mathbb{R}^2$ be a vector field on $\mathbb{R}^2$. What is the definition of $\int_{\mathrm{p}} \mathbf{V}\cdot ds$?

   $$\int_{\mathrm{p}} \mathbf{V}\cdot ds = \int\limits_a^b \mathbf{V}(\mathbf{p}(t))\cdot \mathbf{p}'(t)dt$$

   (c) (1 point) Let $S\subset\mathbb{R}^3$ be a surface parametrized by a one-to-one function $\Phi\colon D\to S$ such that $D\subset\mathbb{R}^2$ is a domain with (u,v) coordinates. Write an integral over D that calculates the area of S.

   (Area of S) $= \iint\limits_D ||T_u\times T_v||dvdu$, where $T_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})$ and $T_v = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v})$.

   (d) (1 point) Let $S\subset\mathbb{R}^3$ be a surface parametrized by a one-to-one function $\Phi\colon D\to\mathbb{R}^3$ such that $D\subset\mathbb{R}^2$ is a domain with (u,v) coordinates. Let $\mathbf{F}\colon\mathbb{R}^3\to\mathbb{R}^3$ be a vector field on $\mathbb{R}^3$. What is the definition of $\iint\limits_\Phi \mathbf{F}\cdot dA$?

   $$\iint\limits_\Phi \mathbf{F}\cdot dA = \iint\limits_D \mathbf{F}(\Phi(u,v))\cdot(T_u\times T_v)dvdu.$$

2. Consider the helix $C \subset \mathbb{R}^3$ which is parametrized by the path

$$\mathbf{p}\colon [0, 2\pi] \to \mathbb{R}^3 \text{ defined by } \mathbf{p}(t) = (\cos(t), \sin(t), t).$$

(a) (2 points) Evaluate the integral

$$\int_C 4 ds.$$

Answer. $\int_C 4ds = \int_0^{2\pi} 4||\mathbf{p}'(t)||dt = \int_0^{2\pi} 4\sqrt{(-\sin(t))^2 + (\cos(t)^2 + 1^2}dt = 8\pi\sqrt{2}$

(b) (2 points) Let $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by $\mathbf{F}(x, y, z) = (0, 0, z)$. Evaluate the line integral

$$\int_C \mathbf{F} \cdot ds.$$

Answer. $\int_C \mathbf{F} \cdot ds = \int_0^{2\pi} (0, 0, t) \cdot (-\sin(t), \cos(t), 1)dt = \int_0^{2\pi} tdt = 2\pi^2.$

3. Let S be the quarter-cylinder with radius $\sqrt{2}$ defined by the equations

$$x^2 + y^2 = 2, x \geq 0, y \geq 0, \text{ and } 0 \leq z \leq 10.$$

(a) (1 point) Write down a parametrization $\Phi\colon D \to S$ for some domain $D \subset \mathbb{R}^2$.
Answer. S can be parametrized by $\Phi\colon D = [0, \pi/2] \times [0, 10] \to \mathbb{R}^3$ defined by

$$\Phi(u, v) = (\sqrt{2}\cos(u), \sqrt{2}\sin(u), v).$$

(b) (2 points) Using any method, write down an equation for the tangent plane to S at the point $(1, 1, 3)$.
Answer. The equation for the tangent plane is $x + y = 2$ (work omitted).

(c) (3 points) Integrate the function $f(x, y, z) = x$ over S. In other words, evaluate the integral

$$\iint_S x dA.$$

Answer. We need to calculate T_u, T_v, and $T_u \times T_v$.

$$T_u = (-\sqrt{2}\sin(u), \sqrt{2}\cos(u), 0), \text{ and } T_v = (0, 0, 1).$$

Thus,

$$T_u \times T_v = (\sqrt{2}\cos(u), \sqrt{2}\sin(u), 0).$$

Then we get $\iint_S x dA = \int_0^{\pi/2} \int_0^{10} \sqrt{2}\cos(u) ||T_u \times T_v|| dv du = \int_0^{\pi/2} \int_0^{10} \sqrt{2}\cos(u)\sqrt{2} dv du = 20$.

Name & Student ID: _____

4. Let S be the surface described by the equations $z = x^2 + y$, $0 \le x \le 1$, and $0 \le y \le x$.

   (a) (1 point) Write down a parametrization of S; $\Phi\colon D \to S$ where $D \subset \mathbb{R}^2$ is the triangle $0 \le x \le 1$ and $0 \le y \le x$.
   Answer. S can be parametrized by $\Phi(x, y) = (x, y, x^2 + y)$ where $(x, y) \in D$.

   (b) (4 points) Find the area of the surface S.
   Answer. Area $= \iint_D ||T_x \times T_y|| dydx = \iint\limits_D ||(1, 0, 2x) \times (0, 1, 1)|| dydx = \iint\limits_D ||(-2x, -1, 1)|| dydx =$
   $\iint\limits_D \sqrt{4x^2 + 2} dydx = \int\limits_0^1 \int\limits_0^x \sqrt{4x^2 + 2} dydx = \int\limits_0^1 x\sqrt{4x^2 + 2} dx = \left(\frac{1}{12}(4x^2 + 2)^{3/2}\right|_{x=0}^1$
   $= \frac{1}{12}(6^{3/2} - 2^{3/2})$

   (c) (2 points) Let $\mathbf{F}(x, y, z) = (x, 0, 0)$ and let Φ be the parametrization from part (a). Evaluate the integral:
   $$\iint\limits_\Phi \mathbf{F} \cdot dA$$
   Answer. $\iint\limits_\Phi \mathbf{F} \cdot dA = \iint_D \mathbf{F}(x, y, x^2 + y) \cdot (-2x, -1, 1) dydx$
   $= \iint_D (x, 0, 0) \cdot (-2x, -1, 1) dydx = \iint_D -2x^2 dydx = \int\limits_0^1 \int\limits_0^x -2x^2 dydx$
   $= \int\limits_0^1 -2x^3 dx = -1/2.$

Name & Student ID: _____

Scratch Paper.