HOMEWORK 4 - MATH 100A - DUE FRIDAY OCTOBER 27TH

Problem 1. (From Herstein §2.5 #1) Determine in each of the parts if the given mapping is a homomorphism. If so, identify its kernel and whether or not the mapping is 1-1 or onto.

- $G = \mathbb{Z}$ under $+$, $G' = \mathbb{Z}_n$. $\varphi(a) = [a]$ for $a \in \mathbb{Z}$.
- $G$ a group, $\varphi: G \to G$ defined by $\varphi(a) = a^{-1}$ for $a \in G$.
- $G$ an abelian group, $\varphi: G \to G$ defined by $\varphi(a) = a^{-1}$ for $a \in G$.
- $G$ the group of all nonzero real numbers under multiplication, $G' = \{1, -1\}$, $\varphi(r) = 1$ if $r > 0$ and $\varphi(r) = -1$ if $r$ negative.
- $G$ an abelian group, $n > 1$ a fixed integer, and $\varphi: G \to G$ defined by $\phi(a) = a^n$ for all $a \in G$.

Problem 2. (Herstein §2.5 #6) Prove that if $\varphi: G \to G'$ is a homomorphism, then $\varphi(G)$ (the image of $G$), is a subgroup of $G$.

Problem 3. (From Herstein §2.5 #7) Show that $\varphi: G \to G'$ is a monomorphism if and only if $\text{Ker}(\varphi) = \{e\}$.

Problem 4. (From Herstein §2.5 #14) If $G$ is abelian and $\varphi: G \to G'$ is a surjective homomorphism then $G'$ is abelian.

Problem 5. (From Herstein §2.5 #24 parts (a) and (e)) If $G_1, G_2$ are two groups, let $G = G_1 \times G_2$ be the Cartesian product of the sets $G_1$ and $G_2$. Define a product in $G$ by $(a_1, b_1) \ast (a_2, b_2) = (a_1 \ast a_2, b_1 \ast b_2)$.

- Prove that $G$ is a group.
- Prove that $G_1 \times G_2 \cong G_2 \times G_1$.

Problem 6. (From Herstein §2.5 #26) If $G$ is a group and $a \in G$, define $\sigma_a: G \to G$ by $\sigma_a(g) = a^g a^{-1}$. We saw in Example 9 of this section that $\sigma_a$ is an isomorphism of $G$ onto itself, so $\sigma_a \in A(G)$, the group all 1-1 mappings of $G$ (as a set) onto itself. Define $\psi: G \to A(G)$ by $\psi(a) = \sigma_a$ for all $a \in G$. Prove that:

- $\psi$ is a homomorphism of $G$ into $A(G)$.
- $\text{Ker}(\psi) = Z(G)$ (the center of $G$).

Problem 7. (From Herstein §2.5 #27) If $\theta$ is an automorphism of $G$ and $N \trianglelefteq G$, prove that $\theta(N) \trianglelefteq G$.

Problem 8. Prove that $S_3 \cong D_6$. Challenge Problem (=not required): prove that any nonabelian group of order 6 is isomorphic to $S_3$.

Problem 9. Let $T \subset \mathbb{R}^3$ denote a regular tetrahedron centered at $(0,0,0) \in \mathbb{R}^3$. For example, you can choose the vertices of $T$ to be at the coordinates $\{(1,1,-1), (1,-1,1), (-1,1,1), (-1,-1,-1)\}$.

Define:

$$\text{Isom}(T) = \{ A \in \text{GL}_3(\mathbb{R}) | A(T) = T \} = \{ \text{matrices which give a bijection: } A: T \to T \} \subset \text{GL}_3(\mathbb{R})$$

Prove that $\text{Isom}(T)$ is a group, compute its order. (You can use the following fact without proof: any matrix $A$ which maps $T$ bijectively onto $T$ must send the vertices to the vertices.)

Problem 10. Prove that $\text{Isom}(T) \cong S_4$. 

A tetrahedron.